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Differently implicational α-universal triple I restriction method of (1, 2, 2) type

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Abstract: From the viewpoints of both fuzzy system and fuzzy reasoning, a new fuzzy reasoning method which contains the α-triple I restriction method as its particular case is proposed. The previous α-triple I restriction principles are improved, and then the optimal restriction solutions of this new method are achieved, especially for seven familiar implications. As its special case, the corresponding results of α-triple I restriction method are obtained and improved. Lastly, it is found by examples that this new method is more reasonable than the α-triple I restriction method.

Keywords: fuzzy reasoning, fuzzy system, triple I method, triple I restriction method, compositional rule of inference method.

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1. Introduction

At present, fuzzy reasoning is widely used in the fields of fuzzy control, complex system modeling and simulation, natural language processing together with affective computing [1–6]. Its basic problems are fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) as follows:

FMP : from given rule $A \to B$ and input A^*, calculate B^* (output) (1)

FMT : from given rule $A \to B$ and input B^*, calculate A^* (output) (2)

in which $A, A^* \in F(U), B, B^* \in F(V)$. $F(U)$ and $F(V)$ denote the set of all fuzzy subsets of universe U and V, respectively.

In order to deal with these problems, the widely used method is the compositional rule of inference (CRI) method proposed by Zadeh [7–11]. In 1999, Wang drew the conclusion that the CRI method had some blemishes, and he put forward the triple I method (as an improvement of the CRI method) [12]. The basic idea of triple I method (taking the α-triple I method as an example) is to seek out the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) making

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \geqslant \alpha \quad (3)$$

hold for any $u \in U, v \in V$, where $\alpha \in [0,1]$ and $\to$ is an implication (see Definition 1).

The triple I method is presently a subject of intensive study. Wang et al. systemically researched the triple I method, as well as its related theories of sustaining degrees and reversibility properties [13–15]. Wang and Pei presented the regular implication which was derived from the left-continuous t-norm, and based on it established the unified forms of triple I method [16,17]. Song et al. analyzed a similar form to (3), i.e., $(A^*(u) \to B^*(v)) \to (A(u) \to B(v)) \geqslant \alpha$, and brought forward the reverse triple I method [18] which was also discussed by [19,20]. Pei formalized related triple I methods and their reversibility properties from a new first-order formal system K^* together with its extension K^*_{ms}, then put fuzzy reasoning into the framework of fuzzy logic [21]. Liu and Wang proposed the concept of pointwise sustaining degrees, and generalized the α-triple I method to the triple I method based on pointwise sustaining degrees [22]. From the syntactical viewpoint [23], Zhang and Yang discussed the triple I method by generalized roots in four familiar logic

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systems [24]. It is shown that the triple I method possesses many acknowledged advantages which are embodied as excellent logic basis, reversibility properties, the property of pointwise optimization and so on [13,15,25,26].

Song et al. indicated in [27] that the formula which was opposite to (3) should also be take into account for a certain kind of fuzzy reasoning, then proposed the α-triple I restriction method. Its solution is the largest $B^* \in F(V)$ (or the smallest $A^* \in F(U)$) satisfying

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \leqslant \alpha \qquad (4)$$

for any $u \in U, v \in V$, where $\alpha \in (0,1)$. Then the α-triple I restriction method was also investigated by [28–30] aiming at (4) or other similar forms.

However, it is found that the triple I method is imperfect in virtue of its inferior response ability and practicability from the viewpoint of some kinds of fuzzy systems [26,31–33]. For example, Li et al. drew the conclusion that 12 fuzzy systems can be practicable in 23 ones based on the CRI method (by analyzing their response ability) [31], while only two usable fuzzy systems are obtained in 51 ones based on the triple I method [32]. Such inferior effect will keep back the development and application of the triple I method.

To solve this problem, enlightened by [26], we generalized the triple I method to the differently implicational universal triple I method of (1, 2, 2) type (universal triple I method for short) in [34]. The idea of universal triple I method is to find the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) such that

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \geqslant \alpha \qquad (5)$$

holds for any $u \in U, v \in V$, where $\alpha \in [0,1]$ and implications $\to_1$ and $\to_2$ can be different.

Similar to the α-triple I restriction method, we also need to investigate the condition opposite to (5), i.e.,

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \leqslant \alpha \qquad (6)$$

where $\alpha \in (0,1)$. The α-triple I restriction method derived from (6) is called the differently implicational α-universal triple I restriction method of (1, 2, 2) type (α-universal triple I restriction method for short). The aim of this paper is to systematically investigate the α-universal triple I restriction method.

2. The α-universal triple I restriction method for FMP

For convenience, we denote $R_1(u,v) = A(u) \to_1 B(v)$, and $x' = 1 - x$.

Definition 1 An implication on $[0,1]$ is a mapping $I : [0,1]^2 \to [0,1]$ satisfying the conditions $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$. $I(a,b)$ is also written as $a \to b$ for any $a, b \in [0,1]$.

Here seven familiar implications are mainly considered, which are Lukasiewicz implication I_L, Goguen implication I_{Go}, Gödel implication I_G, implication I_0, Kleene-Dienes implication I_{KD}, Reichenbach implication I_R, and Gaines-Rescher implication I_{GR} as follows:

$$I_L(a,b) = \begin{cases} 1, & a \leqslant b \\ a' + b, & a > b \end{cases}$$

$$I_{Go}(a,b) = \begin{cases} 1, & a = 0 \\ (b/a) \wedge 1, & a \neq 0 \end{cases}$$

$$I_0(a,b) = \begin{cases} 1, & a \leqslant b \\ a' \vee b, & a > b \end{cases}$$

$$I_G(a,b) = \begin{cases} 1, & a \leqslant b \\ b, & a > b \end{cases}$$

$$I_{KD}(a,b) = a' \vee b$$

$$I_R(a,b) = a' + a \times b$$

$$I_{GR}(a,b) = \begin{cases} 1, & a \leqslant b \\ 0, & a > b \end{cases}.$$

Definition 2 Suppose that Z is any nonempty set, and $F(Z)$ is the set of all fuzzy subsets on Z. Define partial order relation $\leqslant_F$ on $F(Z)$ as $A \leqslant_F B$, if and only if $A(z_0) \leqslant B(z_0)$ for $\forall z_0 \in Z$, where $A, B \in F(Z)$.

Lemma 1 [35] $< F(Z), \leqslant_F >$ is a complete lattice.

For the FMP problem (1), from the viewpoint of α-universal triple I restriction method, we can obtain the following principle:

α-universal triple I restriction principle for FMP

The conclusion B^* (in $< F(V), \leqslant_F >$) of FMP problem (1) is the largest fuzzy set satisfying (6).

Such principle obviously improves the previous α-triple I restriction principle for FMP in [27,28].

Definition 3 Let $A, A^* \in F(U)$, $B \in F(V)$, if B^* (in $< F(V), \leqslant_F >$) makes (6) hold for any $u \in U, v \in V$, then B^* is called an α-FMP-universal triple I restriction solution (α-FMP-solution for short).

Definition 4 Suppose that $A, A^* \in F(U)$, $B \in F(V)$, and nonempty set E is the set of all α-FMP-solutions, and finally that D^* (in $< F(V), \leqslant_F >$) is the supremum of E, then D^* is called an α-SupP-quasi universal triple I restriction solution (α-SupP-quasi solution for short). And if D^* is the maximum of E, then D^* is also called an α-MaxP-universal triple I restriction solution (α-MaxP-solution for short).

Proposition 1 If the implication $\to_2$ satisfies (C1) $a \to b$ is non-decreasing with regard to (w.r.t.) b $(a, b \in [0,1])$, and D_1 is an α-FMP-solution, and finally $D_2 \leqslant_F$

D_1 (in which $D_1, D_2 \in \ < F(V), \leqslant_F >$). Then D_2 is an α-FMP-solution.

Proof Since D_1 is an α-FMP-solution, it follows that $R_1(u,v) \to_2 (A^*(u) \to_2 D_1(v)) \leqslant \alpha$ holds for any $u \in U, v \in V$. Because $D_2 \leqslant_F D_1$ and $\to_2$ satisfies (C1), we get that $A^*(u) \to_2 D_2(v) \leqslant A^*(u) \to_2 D_1(v)$ and

$$\alpha \geqslant R_1(u,v) \to_2 (A^*(u) \to_2 D_1(v)) \geqslant$$

$$R_1(u,v) \to_2 (A^*(u) \to_2 D_2(v))$$

hold for any $u \in U, v \in V$. Therefore D_2 is also an α-FMP-solution. □

Theorem 1 Let the implication $\to_2$ satisfy (C1), $\alpha \in (0,1)$. Then there exists an α-FMP-solution if and only if the following inequality holds for any $u \in U, v \in V$:

$$R_1(u,v) \to_2 (A^*(u) \to_2 0) \leqslant \alpha. \quad (7)$$

Proof (i) If (7) holds, then we take $B^*(v) \equiv 0$ $(v \in V)$, thus B^* obviously satisfies (6), and hence B^* is an α-FMP-solution.

(ii) If there exists a $B^* \in F(V)$ which is an α-FMP-solution, then it follows from Proposition 1 that $B^*(v) \equiv 0$ $(v \in V)$ is also an α-FMP-solution (since $\to_2$ satisfies (C1) and $0 \leqslant_F B^*$), which means (7) holds. □

Remark 1 Suppose that $\to_2$ satisfies (C1) and (7) holds. For an α-FMP-solution B^*, every fuzzy set D which is less than B^*, will be an α-FMP-solution (where B^*, D in $< F(V), \leqslant_F >$). This means that there are many α-FMP-solutions which include $B^*(v) \equiv 0$ $(v \in V)$. This last is a special solution, since (6) always holds no matter what major premise $A \to_1 B$ and minor premise A^* are employed. Therefore, if the optimal α-FMP-solution exists, then it should be the largest one; in other words, it should be the supremum of all solutions (i.e., the supremum of E).

It is easy to get Proposition 2.

Proposition 2 Equation (7) is respectively equivalent to the following formulas:

(i) $2 - R_1(u,v) - A^*(u) \leqslant \alpha$, if $\to_2$ takes I_L;

(ii) $R_1(u,v) \times A^*(u) > 0$, if $\to_2$ takes I_{Go};

(iii) $R_1(u,v) > (A^*(u))'$ and $(R_1(u,v))' \vee (A^*(u))' \leqslant \alpha$, if $\to_2$ takes I_0;

(iv) $R_1(u,v) \times A^*(u) > 0$, if $\to_2$ takes I_G;

(v) $(R_1(u,v))' \vee (A^*(u))' \leqslant \alpha$, if $\to_2$ takes I_{KD};

(vi) $1 - R_1(u,v) \times A^*(u) \leqslant \alpha$, if $\to_2$ takes I_R;

(vii) $R_1(u,v) \times A^*(u) > 0$, if $\to_2$ takes I_{GR}.

It follows from Lemma 1 that $< F(V), \leqslant_F >$ is a complete lattice. Once there exists an α-FMP-solution, then the α-SupP-quasi solution (i.e., the supremum of E) uniquely exists because the nonempty set $E \subset F(V)$.

Theorem 2 If the implication $\to_2$ satisfies (C1) and (C2) $a \to b$ is left-continuous w.r.t. b $(a \in [0,1], b \in (0,1])$, $\alpha \in (0,1)$, and (7) holds. Then the α-SupP-quasi solution is the α-MaxP-solution.

Proof Noting that the α-SupP-quasi solution $B^* = \sup E$, it is enough to prove that B^* is the maximum of E. Consider that

$$E = \{D^* \in F(V) |\, R_1(u,v) \to_2 (A^*(u) \to_2 D^*(v)) \leqslant \alpha,$$

$$u \in U, v \in V\}.$$

On the contrary, assume that $B^* \notin E$, then there exist fuzzy sets $B_1, B_2, \ldots$ in E such that

$$\lim_{n\to\infty} B_n(v) = B^*(v), \ \ v \in V. \quad (8)$$

Since $B^* = \sup E$, we get $B_n(v) \leqslant B^*(v)$ $(v \in V,\ n = 1,2,\ldots)$, and thus it follows from (8) that $B^*(v)$ is the left limit of $\{B_n(v) |\ n = 1,2,\ldots\}$ $(v \in V)$. Notice that $\to_2$ satisfies (C2), so we obtain

$$\lim_{n\to\infty} \{A^*(u) \to_2 B_n(v)\} = A^*(u) \to_2 B^*(v),$$

$$u \in U, v \in V. \quad (9)$$

Because $\to_2$ satisfies (C1), we have $A^*(u) \to_2 B_n(v) \leqslant A^*(u) \to_2 B^*(v)$ $(u \in U, v \in V,\ n = 1,2,\ldots)$, and it follows from (9) that $A^*(u) \to_2 B^*(v)$ is the left limit of $\{A^*(u) \to_2 B_n(v) |\ n = 1,2,\ldots\}$.

Since $B_1, B_2, \ldots \in E$, it follows that

$$R_1(u,v) \to_2 (A^*(u) \to_2 B_n(v)) \leqslant \alpha.$$

Noting that $\to_2$ satisfies (C2), we have

$$\alpha \geqslant \lim_{n\to\infty} \{R_1(u,v) \to_2 (A^*(u) \to_2 B_n(v))\} =$$

$$R_1(u,v) \to_2 (A^*(u) \to_2 B^*(v))$$

which contradicts the assumption. Therefore $B^* \in E$ and thus B^* is the maximum of E. □

Theorem 3 If $\to_2$ takes I_L, $\alpha \in (0,1)$, and (7) holds, then the α-MaxP-solution can be computed as follows:

$$B^*(v) = \alpha - 2 + \inf_{u\in U}\{A^*(u) + R_1(u,v)\}, \ \ v \in V. \quad (10)$$

Proof Let $G_1 = \{v \in V |\ B^*(v) = 0\}$, and $G_2 = \{v \in V |\ B^*(v) > 0\}$. Suppose that $C \in F(V)$, and that $C(v) = 0$ for $v \in G_1$, and that $C(v) < B^*(v)$ for $v \in G_2$. We shall show that C is an α-FMP-solution, that is, the following inequality holds for any $u \in U, v \in V$:

$$R_1(u,v) \to_2 (A^*(u) \to_2 C(v)) \leqslant \alpha. \quad (11)$$

If $v \in G_1$, then it follows from (7) that $C(v) = 0$ satisfies (11) for any $u \in U$.

If $v \in G_2$, then it follows from (10) and $C(v) < B^*(v)$ that the following formula holds for any $u \in U$:

$$C(v) < \alpha - 2 + A^*(u) + R_1(u, v).$$

Thus we get $A^*(u) > C(v)$, and

$$A^*(u) \to_2 C(v) = 1 - A^*(u) + C(v) <$$
$$1 - A^*(u) + \alpha - 2 + A^*(u) + R_1(u, v) =$$
$$\alpha - 1 + R_1(u, v) < R_1(u, v),$$

and then

$$R_1(u, v) \to_2 (A^*(u) \to_2 C(v)) = 1 - R_1(u, v) +$$
$$1 - A^*(u) + C(v) < 1 - R_1(u, v) +$$
$$1 - A^*(u) + \alpha - 2 + A^*(u) + R_1(u, v) = \alpha.$$

Therefore (11) holds for any $u \in U, v \in V$, which implies that C is an α-FMP-solution.

Next, we shall check that B^* determined by (10) is the supremum of all α-FMP-solutions. Assume that $D \in F(V)$, and there exists $v_0 \in V$ such that $D(v_0) > B^*(v_0)$. We shall prove that D is not an α-FMP-solution. In fact, it follows from (10) that there exists $u_0 \in U$ such that

$$D(v_0) > \alpha - 2 + A^*(u_0) + R_1(u_0, v_0) \qquad (12)$$

holds. If $R_1(u_0, v_0) \leqslant A^*(u_0) \to_2 D(v_0)$, then

$$R_1(u_0, v_0) \to_2 (A^*(u_0) \to_2 D(v_0)) = 1 > \alpha,$$

otherwise (i.e., $A^*(u_0) > D(v_0)$ and $R_1(u_0, v_0) > 1 - A^*(u_0) + D(v_0)$ hold), we have from (12) that

$$R_1(u_0, v_0) \to_2 (A^*(u_0) \to_2 D(v_0)) =$$
$$R_1(u_0, v_0) \to_2 (1 - A^*(u_0) + D(v_0)) =$$
$$1 - R_1(u_0, v_0) + 1 - A^*(u_0) + D(v_0) >$$
$$1 - R_1(u_0, v_0) + 1 - A^*(u_0) +$$
$$\alpha - 2 + A^*(u_0) + R_1(u_0, v_0) = \alpha.$$

So D is not an α-FMP-solution.

To sum up, B^* determined by (10) is the supremum of all α-FMP-solutions, thus it is the α-SupP-quasi solution.

Moreover, since I_L satisfies (C1) and (C2), it follows from Theorem 2 that B^* determined by (10) is also the α-MaxP-solution. □

Similar to Theorem 3, we can get Theorems 4–6 (noting that I_{Go}, I_{KD}, I_R all satisfy (C1) and (C2)).

Theorem 4 If $\to_2$ takes I_{Go}, $\alpha \in (0, 1)$, and (7) holds, then the α-MaxP-solution can be expressed as

$$B^*(v) = \inf_{u \in U}\{\alpha \times A^*(u) \times R_1(u, v)\}, \;\; v \in V.$$

Theorem 5 If $\to_2$ takes I_{KD}, $\alpha \in (0, 1)$, and (7) holds, then the α-MaxP-solution can be expressed as

$$B^*(v) = \alpha, \;\; v \in V.$$

Theorem 6 If $\to_2$ takes I_R, $\alpha \in (0, 1)$, and (7) holds, then the α-MaxP-solution can be expressed as

$$B^*(v) = \inf_{u \in U}\Big\{1 - \frac{1 - \alpha}{R_1(u, v) \times A^*(u)}\Big\}, \;\; v \in V.$$

Similar to Theorem 3, we can prove Theorem 7 (noting that I_0 does not satisfy (C2)).

Theorem 7 If $\to_2$ takes I_0, $\alpha \in (0, 1)$, and (7) holds, then the α-SupP-quasi solution can be computed as

$$B^*(v) = \inf_{u \in U}\{A^*(u) \wedge R_1(u, v)\} \wedge \alpha, \;\; v \in V. \qquad (13)$$

Theorem 8 If $\to_2$ takes I_0, $\alpha \in (0, 1)$, and (7) holds, then the α-SupP-quasi solution B^* determined by (13) is the α-MaxP-solution if and only if

$$B^*(v) < A^*(u) \wedge R_1(u, v), \;\; u \in U, v \in V. \qquad (14)$$

Proof Since (7) holds, we get from Proposition 2(iii) that the following formulas hold for any $u \in U, v \in V$:

$$R_1(u, v) > (A^*(u))', \quad (R_1(u, v))' \vee (A^*(u))' \leqslant \alpha. \qquad (15)$$

(i) If (14) holds, then $A^*(u) > B^*(v)$, $R_1(u, v) > B^*(v)$. Hence it follows from (15) that

$$A^*(u) \to_2 B^*(v) = (A^*(u))' \vee B^*(v) < R_1(u, v),$$

and (noting that obviously $B^*(v) \leqslant \alpha$)

$$R_1(u, v) \to_2 (A^*(u) \to_2 B^*(v)) =$$
$$(R_1(u, v))' \vee (A^*(u))' \vee B^*(v) \leqslant \alpha$$

hold for any $u \in U, v \in V$. Therefore the α-SupP-quasi solution B^* determined by (13) is an α-FMP-solution, which implies that B^* is the α-MaxP-solution.

(ii) If (14) does not hold, i.e., there exist $u_0 \in U, v_0 \in V$ such that

$$B^*(v_0) \geqslant A^*(u_0) \wedge R_1(u_0, v_0).$$

We have two cases to be considered.

(a) If $B^*(v_0) \geqslant A^*(u_0)$, then

$$R_1(u_0, v_0) \to_2 (A^*(u_0) \to_2 B^*(v_0)) =$$

$$R_1(u_0, v_0) \to_2 1 = 1 > \alpha.$$

(b) If $B^*(v_0) \geqslant R_1(u_0, v_0)$, then noting that $A^*(u_0) \to_2 B^*(v_0) \geqslant (A^*(u_0))' \vee B^*(v_0) \geqslant R_1(u_0, v_0)$, we also get

$$R_1(u_0, v_0) \to_2 (A^*(u_0) \to_2 B^*(v_0)) = 1 > \alpha.$$

Thus the α-SupP-quasi solution B^* determined by (13) is not an α-FMP-solution, and then it is not the α-MaxP-solution. □

We can get Proposition 3 by virtue of Theorem 8.

Proposition 3 If $\to_2$ takes I_0, $\alpha \in (0, 1)$, and (7) holds, then the α-SupP-quasi solution B^* determined by (13) is the α-MaxP-solution if and only if ($u \in U, v \in V$)

$$\inf_{u \in U}\{A^*(u) \wedge R_1(u, v)\} < A^*(u) \wedge R_1(u, v)$$

or

$$\alpha < A^*(u) \wedge R_1(u, v).$$

Theorem 9 If $\to_2$ takes I_G, $\alpha \in (0, 1)$, and (7) holds, then the α-SupP-quasi solution can be computed as

$$B^*(v) = \inf_{u \in U}\{A^*(u) \wedge R_1(u, v)\} \wedge \alpha, \ \ v \in V.$$

Moreover, the α-SupP-quasi solution B^* is the α-MaxP-solution if and only if

$$B^*(v) < A^*(u) \wedge R_1(u, v), \ \ u \in U, v \in V.$$

Theorem 10 If $\to_2$ takes I_{GR}, $\alpha \in (0, 1)$, and (7) holds, then the α-SupP-quasi solution can be computed as

$$B^*(v) = \inf_{u \in U}\{A^*(u)\}, \ \ v \in V.$$

Moreover, the α-SupP-quasi solution B^* is the α-MaxP-solution if and only if

$$B^*(v) < A^*(u), \ \ u \in U, v \in V.$$

When $\to_1 = \to_2$, the α-FMP-solution degenerates into the solution of the α-triple I restriction method for FMP (1) (α-FMP-triple I restriction solution for short). Denote $\to \triangleq \to_1 = \to_2$. Inspecting the results mentioned above, we can similarly get the following definitions and conclusions of the α-triple I restriction method for FMP. We denote $R(u, v) = A(u) \to B(v)$.

Definition 5 Let $A, A^* \in F(U)$, $B \in F(V)$, if B^* (in $< F(V), \leqslant_F >$) makes (4) hold for any $u \in U, v \in V$, then B^* is called an α-FMP-triple I restriction solution.

Definition 6 Suppose that $A, A^* \in F(U)$, $B \in F(V)$, and that nonempty set E° is the set of all α-FMP-triple I restriction solutions, and finally that D^* (in $< F(V), \leqslant_F >$) is the supremum of E°, then D^* is called an α-SupP-quasi triple I restriction solution. And if D^* is the maximum of E°, then D^* is also called an α-MaxP-triple I restriction solution.

Corollary 1 Let the implication $\to$ satisfy (C1), $\alpha \in (0, 1)$. Then there exists an α-FMP-triple I restriction solution if and only if the following inequality holds for any $u \in U, v \in V$:

$$R(u, v) \to (A^*(u) \to 0) \leqslant \alpha. \tag{16}$$

Corollary 2 Equation (16) is respectively equivalent to the following formulas:

(i) $2 - R(u, v) - A^*(u) \leqslant \alpha$, if $\to$ takes I_L;

(ii) $R(u, v) \times A^*(u) > 0$, if $\to$ takes I_{Go};

(iii) $R(u, v) > (A^*(u))'$ and $(R(u, v))' \vee (A^*(u))' \leqslant \alpha$, if $\to$ takes I_0;

(iv) $R(u, v) \times A^*(u) > 0$, if $\to$ takes I_G;

(v) $(R(u, v))' \vee (A^*(u))' \leqslant \alpha$, if $\to$ takes I_{KD};

(vi) $1 - R(u, v) \times A^*(u) \leqslant \alpha$, if $\to$ takes I_R;

(vii) $R(u, v) \times A^*(u) > 0$, if $\to$ takes I_{GR}.

Similarly, once there exists an α-FMP-triple I restriction solution, then the α-SupP-quasi triple I restriction solution uniquely exists.

Corollary 3 If the implication $\to$ satisfies (C1) and (C2), $\alpha \in (0, 1)$, and (16) holds, then the α-SupP-quasi triple I restriction solution is the α-MaxP-triple I restriction solution.

Corollary 4 If $\to$ takes I_L, $\alpha \in (0, 1)$, and (16) holds, then the α-MaxP-triple I restriction solution is $B^*(v) = \alpha - 2 + \inf_{u \in U}\{A^*(u) + R(u, v)\}, \ \ v \in V$.

Corollary 5 If $\to$ takes I_{Go}, $\alpha \in (0, 1)$, and (16) holds, then the α-MaxP-triple I restriction solution is $B^*(v) = \inf_{u \in U}\{\alpha \times A^*(u) \times R(u, v)\}, \ \ v \in V$.

Corollary 6 If $\to$ takes I_{KD}, $\alpha \in (0, 1)$, and (16) holds, then the α-MaxP-triple I restriction solution is $B^*(v) = \alpha, \ \ v \in V$.

Corollary 7 If $\to$ takes I_R, $\alpha \in (0, 1)$, and (16) holds, then the α-MaxP-triple I restriction solution is

$$B^*(v) = \inf_{u \in U}\left\{1 - \frac{1 - \alpha}{R(u, v) \times A^*(u)}\right\}, \ \ v \in V.$$

Corollary 8 If $\to$ takes I_0, $\alpha \in (0, 1)$, and (16) holds, then the α-SupP-quasi triple I restriction solution is

$$B^*(v) = \inf_{u \in U}\{A^*(u) \wedge R(u, v)\} \wedge \alpha, \ \ v \in V. \tag{17}$$

Corollary 9 If $\to$ takes I_0, $\alpha \in (0, 1)$, and (16) holds, then the α-SupP-quasi triple I restriction solution B^* determined by (17) is the α-MaxP-triple I restriction solution if and only if

$$B^*(v) < A^*(u) \wedge R(u,v),\ \ u \in U, v \in V.$$

Corollary 10 If $\to$ takes I_0, $\alpha \in (0,1)$, (16) holds, then the α-SupP-quasi triple I restriction solution B^* determined by (17) is the α-MaxP-triple I restriction solution if and only if $(u \in U, v \in V)$

$$\inf_{u\in U}\{A^*(u) \wedge R(u,v)\} < A^*(u) \wedge R(u,v),$$

or

$$\alpha < A^*(u) \wedge R(u,v).$$

Corollary 11 If $\to$ takes I_G, $\alpha \in (0,1)$, and (16) holds, then the α-SupP-quasi triple I restriction solution is

$$B^*(v) = \inf_{u\in U}\{A^*(u) \wedge R(u,v)\} \wedge \alpha,\ \ v \in V.$$

Moreover, the α-SupP-quasi triple I restriction solution B^* is the α-MaxP-triple I restriction solution if and only if

$$B^*(v) < A^*(u) \wedge R(u,v),\ \ u \in U, v \in V.$$

Corollary 12 If $\to$ takes I_{GR}, $\alpha \in (0,1)$, and (16) holds, then the α-SupP-quasi triple I restriction solution is

$$B^*(v) = \inf_{u\in U}\{A^*(u)\},\ \ v \in V.$$

Moreover, the α-SupP-quasi triple I restriction solution B^* is the α-MaxP-triple I restriction solution if and only if

$$B^*(v) < A^*(u),\ \ u \in U, v \in V.$$

Remark 2 In [27], Song et al. researched the α-triple I restriction method for FMP, which only aimed at I_0. By Theorem 2 in [27], Song et al. provided the existence condition of α-FMP-triple I restriction solutions as follows:

There exists $u_0 \in U$ such that $A^*(u_0) > 0$, and

$$(A^*(u))' < R(u,v), A^*(u) \wedge R(u,v) \geqslant \alpha'$$

hold for any $u \in U, v \in V$.

Note that $A^*(u) \geqslant \alpha'$ implies $A^*(u) > 0$, thus this existence condition is equivalent to

$$(A^*(u))' < R(u,v), \quad (A^*(u))' \vee (R(u,v))' \leqslant \alpha,$$

$$u \in U, \quad v \in V$$

which coincides with Corollary 2(iii) in this paper. Then, the α-SupP-quasi triple I restriction solution is also shown by Theorem 2 in [27], which is the same as Corollary 8 in this paper. Moreover, they obtain the necessary and sufficient condition that the α-SupP-quasi triple I restriction solution is an α-MaxP-triple I restriction solution. It coincides with Corollary 10 in this paper. Notice that Corollary 2(iii), Corollary 10 can be deduced by Corollary 1 and Corollary 9, respectively. Furthermore, all of these conclusions are the special cases of Proposition 2(iii), Theorem 7 and Proposition 3 in this paper.

Remark 3 In [36], Sun et al. discussed the α-triple I restriction method for FMP where only employed I_L. By Theorem 1 in [36], Sun et al. achieved the existence condition of α-FMP-triple I restriction solutions, and the α-MaxP-triple I restriction solution. It is easy to get that these results are the same as the related ones of Corollary 2(i) and Corollary 4 in this paper. Note that Corollary 2(i) can be deduced by Corollary 1, moreover, it is obvious that Theorem 1 in [36] is a special case of Proposition 2(i) and Theorem 3 in this paper.

Remark 4 In [28], Peng investigated the α-triple I restriction method. By Theorem 2.1.1 in [28], Peng drew the conclusion that if $\to$ satisfies (C1), (C2), and

$$R(u,v) \to (A^*(u) \to 0) < \alpha \tag{18}$$

holds for any $u \in U, v \in V$, then the α-MaxP-triple I restriction solution uniquely exists. It should be pointed out that when $R(u,v) \to (A^*(u) \to 0) = \alpha$, Theorem 2.1.1 in [28] also holds (by virtue of Corollary 1 and Corollary 3 in this paper), which implies that (18) can be changed into (16). Moreover, it is easy to find that Theorem 2.1.1 in [28] is a special case of Corollary 3 in this paper.

For a set $D \subset U$, let χ_D denote the characteristic function of D, which is defined as $\chi_D(u) = \begin{cases} 1,\ u \in D \\ 0,\ u \notin D \end{cases}$, and let $D^c = U - D$.

Remark 5 By Theorem 2.1.3 –Theorem 2.1.6 in [28], Peng obtained the α-MaxP-triple I restriction solutions where respectively employed I_{KD}, I_R, I_L, and I_{Go}. These conclusions coincide with Corollary 4−Corollary 7 in this paper (where (18) can be changed into (16), and note that (16) holds). For example, Theorem 2.1.5 in [28] got the fact that the α-MaxP-triple I restriction solution where $\to$ takes I_L is

$$B^*(v) = \inf_{u\in E_v}\{\alpha-2+A^*(u)+R(u,v)\}\chi_{E_v}+\chi_{E_v^c}, v \in V \tag{19}$$

where $E_v = \{u \in U|(R(u,v))' + (A^*(u))' < 1\}$. Since (16) holds, we get from Corollary 2(i) that $2 - R(u,v) - A^*(u) \leqslant \alpha$ $(u \in U, v \in V)$, which implies $R(u,v) + A^*(u) \geqslant 2 - \alpha > 1$ and then $(R(u,v))' + (A^*(u))' < 1$. Thus (19) is equivalent to

$$B^*(v) = \inf_{u\in U}\{\alpha - 2 + A^*(u) + R(u,v)\}, \quad v \in V.$$

Therefore, Theorem 2.1.5 in [28] coincides with Corollary 4 in this paper.

Example 1 Let $U = V = [0, 1]$, $A(u) = (1 - u)/2$, $B(v) = (1 + v)/4$, $A^*(u) = (1 + u)/2$ and $\alpha = 1/2$, where $u \in U, v \in V$. Suppose that $\to_2 = I_{Go}$, $\to_1 = I_L$ in the α-universal triple I restriction method for FMP. We now calculate the α-MaxP-solution.

$$R_1(u,v) = A(u) \to_1 B(v) = I_L(A(u), B(v)) =$$

$$\begin{cases} 1 - \dfrac{1-u}{2} + \dfrac{1+v}{4}, & \dfrac{1-u}{2} > \dfrac{1+v}{4} \\ 1, & \dfrac{1-u}{2} \leqslant \dfrac{1+v}{4} \end{cases} =$$

$$\begin{cases} \dfrac{2u+v+3}{4}, & 2u+v<1 \\ 1, & 2u+v \geqslant 1 \end{cases}.$$

Here (7) obviously holds (from Proposition 2(ii)). Thus we get from Theorem 4 that the α-MaxP-solution ($v \in V$) is

$$B^*(v) = \inf_{u\in U}\{\alpha \times A^*(u) \times R_1(u,v)\} =$$

$$\inf_{u\in[0,1]}\{\alpha \times A^*(u) \times R_1(u,v) | 2u+v<1\} \wedge$$

$$\inf_{u\in[0,1]}\{\alpha \times A^*(u) \times R_1(u,v) | 2u+v \geqslant 1\} =$$

$$\inf_{u\in[0,1]}\left\{\frac{u+1}{4} \times \frac{2u+v+3}{4} | 2u+v<1\right\} \wedge$$

$$\inf_{u\in[0,1]}\left\{\frac{u+1}{4} | 2u+v \geqslant 1\right\}$$

(i) Suppose $v = 1$, then $\{u \in [0,1] \mid 2u + v < 1\} = \varnothing$, and $0 \in \{u \in [0,1] \mid 2u+v \geqslant 1\}$. Taking into account that $\dfrac{u+1}{4}$ is increasing w.r.t. u, we get

$$B^*(v) = (\inf \varnothing) \wedge \frac{1}{4} = 1 \wedge \frac{1}{4} = \frac{1}{4} = \frac{3+v}{16}.$$

(ii) Suppose $0 \leqslant v < 1$, then $0 \in \{u \in [0,1] \mid 2u+v < 1\}$. Since $\dfrac{u+1}{4}, \dfrac{2u+v+3}{4}$ are increasing w.r.t. u, we have

$$B^*(v) = \frac{3+v}{16} \wedge \frac{\frac{1-v}{2}+1}{4} = \frac{3+v}{16} \wedge \frac{3-v}{8} = \frac{3+v}{16}$$

where $\dfrac{3+v}{16} < \dfrac{3-v}{8}$ since $v < 1$.

Together we obtain

$$B^*(v) = \frac{3+v}{16}, \ \ v \in V.$$

Example 2 Let U, V, A, B, A^*, α be the same as in Example 1. Suppose that $\to = I_{Go}$ in the α-triple I restriction method for FMP. We now calculate the α-MaxP-triple I restriction solution.

$$R(u,v) = I_{Go}(A(u), B(v)) = \begin{cases} \dfrac{v+1}{2-2u}, & 2u+v<1 \\ 1, & 2u+v \geqslant 1 \end{cases}$$

Here (16) obviously holds (from Corollary 2(ii)), and it follows from Corollary 5 that the α-MaxP-triple I restriction solution ($v \in V$) is

$$B^*(v) = \inf_{u\in U}\{\alpha \times A^*(u) \times R(u,v)\} =$$

$$\inf_{u\in[0,1]}\{\frac{u+1}{4} \times \frac{v+1}{2-2u} | 2u+v<1\} \wedge$$

$$\inf_{u\in[0,1]}\{\frac{u+1}{4} | 2u+v \geqslant 1\}.$$

(i) Suppose $v = 1$. We can also get $B^*(v) = 1/4 = (1+v)/8$.

(ii) Suppose $0 \leqslant v < 1$, we similarly have $B^*(v) = \dfrac{1+v}{8} \wedge \dfrac{3-v}{8} = \dfrac{1+v}{8}$. Together we obtain

$$B^*(v) = \frac{1+v}{8}, \ \ v \in V.$$

Remark 6 Aiming at the same U, V, A, B, A^*, α, the α-MaxP-solution in Example 1 is larger than the α-MaxP-triple I restriction solution in Example 2 (noting that $v < 1$ implies $\dfrac{3+v}{16} > \dfrac{1+v}{8}$, and that $v = 1$ implies $\dfrac{3+v}{16} = \dfrac{1+v}{8}$). From the basic idea of the α-universal triple I restriction method (i.e., α-universal triple I restriction principle for FMP, which seeks out the largest B^* satisfying (6)), the α-universal triple I restriction method makes the reasoning closer, thus it is better than the α-triple I restriction method.

3. The α-universal triple I restriction method for FMT

For the FMT problem (2), from the viewpoint of α-universal triple I restriction method, we can achieve the following principle (which similarly improves the previous α-triple I restriction principle for FMT in [27,28]): α-universal triple I restriction principle for FMT. The conclusion A^* (in $< F(U), \leqslant_F >$) of FMT problem (2) is the smallest fuzzy set satisfying (6).

Definition 7 Let $A \in F(U)$, $B, B^* \in F(V)$, if A^* (in $< F(U), \leqslant_F >$) makes (6) hold for any $u \in U, v \in V$, then A^* is called an α-FMT-universal triple I restriction solution (α-FMT-solution for short).

Definition 8 Suppose that $A \in F(U)$, $B, B^* \in F(V)$, and that nonempty set F is the set of all α-FMT-solutions, and finally that C^* (in $< F(U), \leqslant_F >$) is the infimum of F, then C^* is called an α-InfT-quasi universal triple I restriction solution (α-InfT-quasi solution for short). And if C^* is the minimum of F, then C^* is also called an α-MinT-universal triple I restriction solution (α-MinT-solution for short).

Similar to Proposition 1 and Theorem 1, we can prove Proposition 4 and Theorem 11.

Proposition 4 If the implication $\rightarrow_2$ satisfies (C1) and (C3) $a \rightarrow b$ is non-increasing w.r.t. a $(a, b \in [0,1])$, and C_1 is an α-FMT-solution, and finally $C_1 \leqslant_F C_2$ (in which $C_1, C_2 \in < F(U), \leqslant_F >$). Then C_2 is an α-FMT-solution.

Theorem 11 Let the implication $\rightarrow_2$ satisfy (C1) and (C3), $\alpha \in (0,1)$. Then there exists an α-FMT-solution if and only if the following inequality holds for any $u \in U, v \in V$:

$$R_1(u,v) \rightarrow_2 (1 \rightarrow_2 B^*(v)) \leqslant \alpha. \tag{20}$$

Remark 7 Suppose that $\rightarrow_2$ satisfies (C1), (C3) and that (20) holds. For an α-FMT-solution A^*, every fuzzy set C which is larger than A^*, will be an α-FMT-solution (where A^*, C in $< F(U), \leqslant_F >$). This implies that there are many α-FMT-solutions including $A^*(u) \equiv 1$ $(u \in U)$. The last is a special solution, because (6) always holds no matter what $A \rightarrow_1 B$ and B^* are adopted. Thus, if the optimal α-FMT-solution exists, then it should be the smallest one; in other words, it should be the infimum of all solutions (i.e., the infimum of F).

It is easy to get Proposition 5.

Proposition 5 Equation (20) is respectively equivalent to the following formulas:

(i) $1 - R_1(u,v) + B^*(v) \leqslant \alpha$, if $\rightarrow_2$ takes I_L;

(ii) $R_1(u,v) > B^*(v)$, $B^*(v) \leqslant \alpha \times R_1(u,v)$, if $\rightarrow_2$ takes I_{Go};

(iii) $R_1(u,v) > B^*(v)$ and $(R_1(u,v))' \vee B^*(v) \leqslant \alpha$, if $\rightarrow_2$ takes I_0;

(iv) $R_1(u,v) > B^*(v)$ and $B^*(v) \leqslant \alpha$, if $\rightarrow_2$ takes I_G;

(v) $(R_1(u,v))' \vee B^*(v) \leqslant \alpha$, if $\rightarrow_2$ takes I_{KD};

(vi) $(R_1(u,v))' + R_1(u,v) \times B^*(v) \leqslant \alpha$, if $\rightarrow_2$ takes I_R;

(vii) $1 > B^*(v)$ and $R_1(u,v) > 0$, if $\rightarrow_2$ takes I_{GR}.

Similarly, once there exists an α-FMT-solution, then the α-InfT-quasi solution (i.e., the infimum of F) uniquely exists since the nonempty set $F \subset F(U)$.

Theorem 12 If the implication $\rightarrow_2$ satisfies (C1), (C2), (C3) and (C4) $a \rightarrow b$ is right-continuous w.r.t. a $(a \in [0,1), b \in [0,1])$, $\alpha \in (0,1)$, and (20) holds. Then the α-InfT-quasi solution is an α-MinT-solution.

Proof Note that the α-InfT-quasi solution $A^* = \inf F$, thus it is enough to verify that A^* is the minimum of F. It is obvious that

$$F = \{C^* \in F(U) | \ R_1(u,v) \rightarrow_2 (C^*(u) \rightarrow_2 B^*(v)) \leqslant \alpha, \quad u \in U, v \in V\}.$$

On the contrary, assume that $A^* \notin F$, then there exist fuzzy sets $A_1, A_2, \ldots$ in F such that

$$\lim_{n\to\infty} A_n(u) = A^*(u), \ \ u \in U. \tag{21}$$

Because $A^* = \inf F$, it follows that $A_n(u) \geqslant A^*(u)$ $(u \in U,\ n = 1, 2, \ldots)$, and thus we get from (21) that $A^*(u)$ is the right limit of $\{A_n(u) | \ n = 1, 2, \ldots\}$ $(u \in U)$. Considering that $\rightarrow_2$ satisfies (C4), we achieve

$$\lim_{n\to\infty} \{A_n(u) \rightarrow_2 B^*(v)\} = A^*(u) \rightarrow_2 B^*(v), \quad u \in U, v \in V \tag{22}$$

Since $\rightarrow_2$ satisfies (C3), we get $A_n(u) \rightarrow_2 B^*(v) \leqslant A^*(u) \rightarrow_2 B^*(v)$ $(u \in U, v \in V,\ n = 1, 2, \ldots)$, and it follows from (22) that $A^*(u) \rightarrow_2 B^*(v)$ is the left limit of $\{A_n(u) \rightarrow_2 B^*(v) | \ n = 1, 2, \ldots\}$.

Because $A_1, A_2, \ldots \in F$, it follows that

$$R_1(u,v) \rightarrow_2 (A_n(u) \rightarrow_2 B^*(v)) \leqslant \alpha, u \in U, v \in V$$

Noting that $\rightarrow_2$ satisfies (C2), we get

$$\alpha \geqslant \lim_{n\to\infty} \{R_1(u,v) \rightarrow_2 (A_n(u) \rightarrow_2 B^*(v))\} = R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)),$$

which contradicts the assumption. So $A^* \in F$ and thus A^* is the minimum of F. □

Theorem 13 If $\rightarrow_2$ takes I_L, $\alpha \in (0,1)$, and (20) holds, then the α-MinT-solution can be computed as

$$A^*(u) = 2 - \alpha + \sup_{v \in V} \{B^*(v) - R_1(u,v)\}, \ \ u \in U. \tag{23}$$

Proof Let $H_1 = \{u \in U | \ A^*(u) = 1\}$ and $H_2 = \{u \in U | \ A^*(u) < 1\}$. Assume that $C \in F(U)$, and that $C(u) = 1$ for $u \in H_1$, and that $C(u) > A^*(u)$ for $u \in H_2$. We shall prove that C is an α-FMT-solution, i.e., the following inequality holds for any $u \in U, v \in V$:

$$R_1(u,v) \rightarrow_2 (C(u) \rightarrow_2 B^*(v)) \leqslant \alpha. \tag{24}$$

If $u \in H_1$, then it follows from (20) that $C(u) = 1$ satisfies (24) for any $v \in V$.

If $u \in H_2$, then it follows from (23) and $C(u) > A^*(u)$ that the following formula holds for any $v \in V$:

$$C(u) > 2 - \alpha + B^*(v) - R_1(u,v).$$

So we have $C(u) > B^*(v)$, and

$$C(u) \to_2 B^*(v) = 1 - C(u) + B^*(v) < 1 - (2 - \alpha + B^*(v) - R_1(u,v)) + B^*(v) = \alpha - 1 + R_1(u,v) < R_1(u,v),$$

and then

$$R_1(u,v) \to_2 (C(u) \to_2 B^*(v)) = 1 - R_1(u,v) + 1 - C(u) + B^*(v) < 1 - R_1(u,v) + 1 - (2 - \alpha + B^*(v) - R_1(u,v)) + B^*(v) = \alpha.$$

Thus (24) holds for any $u \in U, v \in V$, which means that C is an α-FMT-solution.

Next, we shall verify that A^* determined by (23) is the infimum of all α-FMT-solutions. Assume that $D \in F(U)$, and that there exists $u_0 \in U$ such that $D(u_0) < A^*(u_0)$. We shall show that D is not an α-FMT-solution. In fact, it follows from (23) that there exists $v_0 \in V$ such that

$$D(u_0) < 2 - \alpha + B^*(v_0) - R_1(u_0, v_0). \tag{25}$$

If $R_1(u_0, v_0) \leqslant D(u_0) \to_2 B^*(v_0)$, then

$$R_1(u_0, v_0) \to_2 (D(u_0) \to_2 B^*(v_0)) = 1 > \alpha;$$

otherwise (i.e., $D(u_0) > B^*(v_0)$ and $R_1(u_0, v_0) > 1 - D(u_0) + B^*(v_0)$ hold), we get from (25) that

$$R_1(u_0,v_0) \to_2 (D(u_0) \to_2 B^*(v_0)) = 1 - R_1(u_0,v_0) + 1 - D(u_0) + B^*(v_0) > 1 - R_1(u_0,v_0) + 1 - (2 - \alpha + B^*(v_0) - R_1(u_0,v_0)) + B^*(v_0) = \alpha.$$

Therefore D is not an α-FMT-solution.

Summarizing above, A^* determined by (23) is the infimum of all α-FMT-solutions, thus it is the α-InfT-quasi solution.

Moreover, since I_L satisfies (C1), (C2), (C3) and (C4), it follows from Theorem 12 that A^* determined by (23) is also the α-MinT-solution. □

It is similar to Theorem 13 that we can get Theorems 14–16, where I_{KD}, I_R satisfy (C1), (C2), (C3), and (C4), and I_0 only satisfies (C1), (C3), and (C4).

Theorem 14 If $\to_2$ takes I_{KD}, $\alpha \in (0,1)$, and (20) holds, then the α-MinT-solution can be computed as

$$A^*(u) = \alpha', \ \ u \in U.$$

Theorem 15 If $\to_2$ takes I_R, $\alpha \in (0,1)$, and (20) holds, then the α-MinT-solution can be computed as

$$A^*(u) = \sup_{v \in V} \left\{ \frac{1-\alpha}{R_1(u,v) \times (B^*(v))'} \right\}, \ \ u \in U.$$

Theorem 16 If $\to_2$ takes I_0, $\alpha \in (0,1)$, and (20) holds, then the α-InfT-quasi solution can be computed as

$$A^*(u) = \sup_{v \in V} \{(R_1(u,v))' \vee B^*(v)\} \vee \alpha', \ \ u \in U. \tag{26}$$

Theorem 17 If $\to_2$ takes I_0, $\alpha \in (0,1)$, and (20) holds, then the α-InfT-quasi solution A^* determined by (26) is the α-MinT-solution if and only if

$$A^*(u) > (R_1(u,v))' \vee B^*(v), \ \ u \in U, v \in V. \tag{27}$$

Proof Since (20) holds, it follows from Proposition 5(iii) that the following formulas hold for any $u \in U, v \in V$:

$$R_1(u,v) > B^*(v), \quad (R_1(u,v))' \vee B^*(v) \leqslant \alpha. \tag{28}$$

(i) If (27) holds, then we get from (28) that

$$A^*(u) \to_2 B^*(v) = (A^*(u))' \vee B^*(v) < R_1(u,v),$$

and (noting that obviously $A^*(u) \geqslant \alpha'$ and $(A^*(u))' \leqslant \alpha$)

$$R_1(u,v) \to_2 (A^*(u) \to_2 B^*(v)) = (R_1(u,v))' \vee (A^*(u))' \vee B^*(v) \leqslant \alpha$$

hold for any $u \in U, v \in V$. Therefore the α-InfT-quasi solution A^* is an α-FMT-solution, which means that A^* is the α-MinT-solution.

(ii) If (27) does not hold, i.e., there exist $u_0 \in U, v_0 \in V$ such that

$$A^*(u_0) \leqslant (R_1(u_0,v_0))' \vee B^*(v_0).$$

We have two cases to be considered.

(a) If $A^*(u_0) \leqslant B^*(v_0)$, then

$$R_1(u_0,v_0) \to_2 (A^*(u_0) \to_2 B^*(v_0)) = R_1(u_0,v_0) \to_2 1 = 1 > \alpha.$$

(b) If $A^*(u_0) \leqslant (R_1(u_0,v_0))'$, then $A^*(u_0) \to_2 B^*(v_0) \geqslant (A^*(u_0))' \vee B^*(v_0) \geqslant R_1(u_0,v_0)$ and thus

$$R_1(u_0,v_0) \to_2 (A^*(u_0) \to_2 B^*(v_0)) = 1 > \alpha.$$

As a result, the α-InfT-quasi solution A^* is not an α-FMT-solution, and then it is not the α-MinT-solution. □

Similar to Theorems 16 and 17, we can prove Theorems 18–20.

Theorem 18 If $\to_2$ takes I_{Go}, $\alpha \in (0,1)$, and (20) holds, then the α-InfT-quasi solution can be computed as

$$A^*(u) = \sup_{v \in V} \left\{ \frac{B^*(v)}{\alpha \times R_1(u,v)} \right\}, \ \ u \in U. \tag{29}$$

Moreover, if

$$A^*(u) > \frac{B^*(v)}{\alpha \times R_1(u,v)},\ \ u \in U, v \in V \qquad (30)$$

holds, then the α-InfT-quasi solution A^* determined by (29) is the α-MinT-solution.

Remark 8 In Theorem 18, the condition (30) is not the necessary condition making the α-InfT-quasi solution A^* determined by (29) be the α-MinT-solution. In fact, suppose that (30) does not hold, i.e., there exist $u_0 \in U, v_0 \in V$ such that

$$A^*(u_0) \leqslant B^*(v_0)/(\alpha \times R_1(u_0, v_0)).$$

If the cases that $A^*(u_0) = B^*(v_0)/(\alpha \times R_1(u_0, v_0))$ and that $R_1(u_0, v_0) > A^*(u_0) \to_2 B^*(v_0)$ happen, we have

$$R_1(u_0, v_0) \to_2 (A^*(u_0) \to_2 B^*(v_0)) =$$

$$B^*(v_0)/(A^*(u_0) \times R_1(u_0, v_0)) = \alpha,$$

which implies that the α-InfT-quasi solution A^* is the α-MinT-solution (when (30) does not hold).

Theorem 19 If $\to_2$ takes I_G, $\alpha \in (0,1)$, and (20) holds, then the α-InfT-quasi solution can be computed as

$$A^*(u) = \sup_{v \in V}\{B^*(v)\},\ \ u \in U.$$

Moreover, the α-InfT-quasi solution A^* is the α-MinT-solution if and only if

$$A^*(u) > B^*(v),\ \ u \in U, v \in V.$$

Theorem 20 If $\to_2$ takes I_{GR}, $\alpha \in (0,1)$, and (20) holds, then the α-InfT-quasi solution can be computed as

$$A^*(u) = \sup_{v \in V}\{B^*(v)\},\ \ u \in U.$$

Moreover, the α-InfT-quasi solution A^* is the α-MinT-solution if and only if

$$A^*(u) > B^*(v),\ \ u \in U, v \in V.$$

When $\to_1 = \to_2$, the α-FMT-solution degenerates into the solution of α-triple I restriction method for FMT (2) (α-FMT-triple I restriction solution for short). Denote $\to \triangleq \to_1 = \to_2$. From the conclusions mentioned above, we can similarly obtain the following definitions and results of the α-triple I restriction method for FMT.

Definition 9 Let $A \in F(U)$, $B, B^* \in F(V)$, if A^* (in $< F(U), \leqslant_F >$) makes (4) hold for any $u \in U, v \in V$, then A^* is called an α-FMT-triple I restriction solution.

Definition 10 Suppose that $A \in F(U)$, $B, B^* \in F(V)$, and the nonempty set F° is the set of all α-FMT-triple I restriction solutions, and finally that C^* (in $< F(U), \leqslant_F >$) is the infimum of F°, then C^* is called an α-InfT-quasi triple I restriction solution. And if C^* is the minimum of F°, then C^* is also called an α-MinT-triple I restriction solution.

Corollary 13 Let the implication $\to$ satisfy (C1) and (C3), $\alpha \in (0,1)$. Then there exists an α-FMT-triple I restriction solution if and only if the following inequality holds for any $u \in U, v \in V$:

$$R(u,v) \to (1 \to B^*(v)) \leqslant \alpha. \qquad (31)$$

Corollary 14 Equation (31) is respectively equivalent to the following formulas:

(i) $1 - R(u,v) + B^*(v) \leqslant \alpha$, if $\to$ takes I_L;

(ii) $R(u,v) > B^*(v),\ B^*(v) \leqslant \alpha \times R(u,v)$, if $\to$ takes I_{Go};

(iii) $R(u,v) > B^*(v)$ and $(R(u,v))' \vee B^*(v) \leqslant \alpha$, if $\to$ takes I_0;

(iv) $R(u,v) > B^*(v)$ and $B^*(v) \leqslant \alpha$, if $\to$ takes I_G;

(v) $(R(u,v))' \vee B^*(v) \leqslant \alpha$, if $\to$ takes I_{KD};

(vi) $(R(u,v))' + R(u,v) \times B^*(v) \leqslant \alpha$, if $\to$ takes I_R;

(vii) $1 > B^*(v)$ and $R(u,v) > 0$, if $\to$ takes I_{GR}.

Similarly, once there exists an α-FMT-triple I restriction solution, then the α-InfT-quasi triple I restriction solution uniquely exists.

Corollary 15 If the implication $\to$ satisfies (C1), (C2), (C3) and (C4), $\alpha \in (0,1)$, and (31) holds. Then the α-InfT-quasi triple I restriction solution is the α-MinT-triple I restriction solution.

Corollary 16 If $\to$ takes I_L, $\alpha \in (0,1)$, and (31) holds, then the α-MinT-triple I restriction solution is $A^*(u) = 2 - \alpha + \sup_{v \in V}\{B^*(v) - R(u,v)\}$ $(u \in U)$.

Corollary 17 If $\to$ takes I_{KD}, $\alpha \in (0,1)$, and (31) holds, then the α-MinT-triple I restriction solution is $A^*(u) = \alpha'$ $(u \in U)$.

Corollary 18 If $\to$ takes I_R, $\alpha \in (0,1)$, and (31) holds, then the α-MinT-triple I restriction solution is

$$A^*(u) = \sup_{v \in V}\left\{\frac{1-\alpha}{R(u,v) \times (B^*(v))'}\right\},\ \ u \in U.$$

Corollary 19 If $\to$ takes I_0, $\alpha \in (0,1)$, and (31) holds, then the α-InfT-quasi triple I restriction solution is

$$A^*(u) = \sup_{v \in V}\{(R(u,v))' \vee B^*(v)\} \vee \alpha',\ \ u \in U. \qquad (32)$$

Corollary 20 If $\to$ takes I_0, $\alpha \in (0,1)$, and (31) holds, then the α-InfT-quasi triple I restriction solution A^*

determined by (32) is the α-MinT-triple I restriction solution if and only if

$$A^*(u) > (R(u,v))' \vee B^*(v), \;\; u \in U, v \in V. \tag{33}$$

Corollary 21 If $\rightarrow$ takes I_{Go}, $\alpha \in (0,1)$, and (31) holds, then the α-InfT-quasi triple I restriction solution is

$$A^*(u) = \sup_{v \in V} \left\{ \frac{B^*(v)}{\alpha \times R(u,v)} \right\}, \;\; u \in U.$$

Moreover, if

$$A^*(u) > \frac{B^*(v)}{\alpha \times R(u,v)}, \;\; u \in U, v \in V$$

holds, then the α-InfT-quasi triple I restriction solution A^* is the α-MinT-triple I restriction solution.

Corollary 22 If $\rightarrow$ takes I_G, $\alpha \in (0,1)$, and (31) holds, then the α-InfT-quasi triple I restriction solution is

$$A^*(u) = \sup_{v \in V} \{B^*(v)\}, \;\; u \in U.$$

Moreover, the α-InfT-quasi triple I restriction solution A^* is the α-MinT-triple I restriction solution if and only if

$$A^*(u) > B^*(v), \;\; u \in U, v \in V.$$

Corollary 23 If $\rightarrow$ takes I_{GR}, $\alpha \in (0,1)$, and (31) holds, then the α-InfT-quasi triple I restriction solution is

$$A^*(u) = \sup_{v \in V} \{B^*(v)\}, \;\; u \in U.$$

Moreover, the α-InfT-quasi triple I restriction solution A^* is the α-MinT-triple I restriction solution if and only if

$$A^*(u) > B^*(v), \;\; u \in U, v \in V.$$

Remark 9 In [27], Song et al. researched the α-triple I restriction method for FMT, which only employed I_0. By Theorem 3 in [27], Song et al. have given the existence condition of α-FMT-triple I restriction solutions as follows:

There exists $v_0 \in V$ such that $B^*(v_0) < 1$, and

$$R(u,v) > B^*(v), \quad (R(u,v))' \vee B^*(v) \leqslant \alpha$$

hold for any $u \in U, v \in V$.

Note that $R(u,v) > B^*(v)$ implies $1 > B^*(v)$, therefore this existence condition is equivalent to

$$R(u,v) > B^*(v), \quad (R(u,v))' \vee B^*(v) \leqslant \alpha$$

which is the same as Corollary 14(iii). Thus it is a special case of Corollary 13. Then, the α-InfT-quasi triple I restriction solution is also shown by Theorem 3 in [27], which coincides with Corollary 19. Moreover, they also achieved the necessary and sufficient condition that the α-InfT-quasi triple I restriction solution was the α-MinT-triple I restriction solution. It is similar to Remark 2 that this is a special case of Corollary 20 in this paper. Furthermore, all of these results are the special cases of Proposition 5(iii), Theorem 16 and Theorem 17 in this paper.

Remark 10 In [36], Sun et al. researched the α-triple I restriction method for FMT where only aimed at I_L. By Theorem 2 in [36], Sun et al. obtained the existence condition of α-FMT-triple I restriction solutions, and the α-MinT-triple I restriction solution. It is easy to know that these conclusions coincide with the related ones of Corollary 14(i) and Corollary 16 in this paper. Taking into account that Corollary 14(i) can be deduced by Corollary 13, moreover it is evident that Theorem 2 in [36] is a special case of Proposition 5 and Theorem 13 in this paper.

Remark 11 By Theorem 2.2.1 in [28], Peng obtained the result that if $\rightarrow$ satisfies (C1), (C2), (C3), and (C4), and

$$R(u,v) \rightarrow (1 \rightarrow B^*(v)) < \alpha \tag{34}$$

held for any $u \in U, v \in V$, then the α-MinT-triple I restriction solution uniquely exists. It should be pointed out that when $R(u,v) \rightarrow (1 \rightarrow B^*(v)) = \alpha$, Theorem 2.2.1 in [28] also holds (from Corollary 13 and Corollary 15 in this paper), which means that (34) can be transformed into (31). Furthermore, it is easy to know that Theorem 2.2.1 in [28] is a special case of Corollary 15 in this paper.

Remark 12 By Theorems 2.2.3 and 2.2.4 in [28], Peng achieved the α-MinT-triple I restriction solutions where respectively employed I_R, I_L. These results coincide with Corollaries 16 and 18 in this paper (where (34) can be transformed into (31), and notice that (31) holds). For example, Theorem 2.2.4 in [28] drew the conclusion that the α-MinT-triple I restriction solution (where $\rightarrow$ takes I_L) is

$$A^*(u) = \sup_{v \in E_u} \{\alpha' + B^*(v) + (R(u,v))'\}, \quad u \in U \tag{35}$$

where $E_u = \{v \in V | B^*(v) < R(u,v)\}$. Since (31) holds, it follows from Corollary 14(i) that $1 - R(u,v) + B^*(v) \leqslant \alpha$ ($u \in U, v \in V$), which implies

$$B^*(v) \leqslant R(u,v) + \alpha - 1 < R(u,v).$$

So (35) is equivalent to

$$A^*(u) = \sup_{v \in V} \{\alpha' + B^*(v) + (R(u,v))'\} =$$

$$2-\alpha+\sup_{v\in V}\{B^*(v)-R(u,v)\},\quad u\in U.$$

Thus Theorem 2.2.4 in [28] coincides with Corollary 16 in this paper.

Remark 13 In [28], Peng pointed out the fact that the α-MinT-triple I restriction solution for I_{Go} may not exist, and he did not give corresponding results w.r.t. the α-InfT-quasi triple I restriction solution and the α-MinT-triple I restriction solution. However, by Corollary 21 in this paper, we obtain the expression of the α-InfT-quasi triple I restriction solution, and the condition that the α-InfT-quasi triple I restriction solution is the α-MinT-triple I restriction solution (where $\rightarrow$ takes I_{Go}). Therefore we provide the further results related to I_{Go}.

Example 3 Let $U=V=[0,1]$, $A(u)=(1-u)/2$, $B(v)=(1+v)/4$, $B^*(v)=(1-v)/2$ and $\alpha=3/4$, where $u\in U, v\in V$. Suppose that $\rightarrow_2=I_R$, $\rightarrow_1=I_L$ in the α-universal triple I restriction method for FMT. We now calculate the α-MinT-solution:

$$R_1(u,v)=I_L(A(u),B(v))=$$

$$\begin{cases}\dfrac{2u+v+3}{4}, & 2u+v<1\\ 1, & 2u+v\geqslant 1\end{cases}.$$

For (20), we have

$$(R_1(u,v))'+R_1(u,v)\times B^*(v)=$$

$$\begin{cases}1-\dfrac{2u+v+3}{4}+\dfrac{2u+v+3}{4}\times\dfrac{1-v}{2}, & 2u+v<1\\ 0+1\times\dfrac{1-v}{2}, & 2u+v\geqslant 1\end{cases}=$$

$$\begin{cases}\dfrac{5-2u-v}{8}-\dfrac{v(2u+v+3)}{8}, & 2u+v<1\\ \dfrac{1-v}{2}, & 2u+v\geqslant 1\end{cases}\leqslant\alpha.$$

Thus (20) holds (from Proposition 5(vi)). Then we get from Theorem 15 that the α-MinT-solution is

$$A^*(u)=\sup_{v\in V}\{\alpha'/[R_1(u,v)\times(B^*(v))']\}=$$

$$\sup_{v\in[0,1]}\{\alpha'/[R_1(u,v)\times(B^*(v))']\mid 2u+v<1\}\vee$$

$$\sup_{v\in[0,1]}\{\alpha'/[R_1(u,v)\times(B^*(v))']\mid 2u+v\geqslant 1\}=$$

$$\sup_{v\in[0,1]}\{2/[(2u+v+3)\times(1+v)]\mid 2u+v<1\}\vee$$

$$\sup_{v\in[0,1]}\{1/(2+2v)\mid 2u+v\geqslant 1\},\quad u\in U.$$

(i) Suppose $1\geqslant u\geqslant 1/2$, then $\{v\in[0,1]\mid 2u+v<1\}=\varnothing$, and $0\in\{v\in[0,1]\mid 2u+v\geqslant 1\}$. Taking into account that $\dfrac{1}{1+v}$ is decreasing w.r.t. v, we get

$$A^*(u)=(\sup\varnothing)\vee\frac{1}{2}=0\vee\frac{1}{2}=\frac{1}{2}.$$

(ii) Suppose $1/2>u\geqslant 0$, then $0\in\{v\in[0,1]\mid 2u+v<1\}$. Noting that $\dfrac{1}{1+v},\dfrac{1}{2u+v+3}$ are decreasing w.r.t. v, we have

$$A^*(u)=\frac{2}{2u+3}\vee\frac{1}{2+2(1-2u)}=$$

$$\frac{2}{2u+3}\vee\frac{1}{4-4u}=\frac{2}{2u+3}$$

where $\dfrac{2}{2u+3}>\dfrac{1}{4-4u}$ since $u<\dfrac{1}{2}$.

Together we obtain

$$A^*(u)=\begin{cases}2/(2u+3), & 1/2>u\geqslant 0\\ 1/2, & 1\geqslant u\geqslant 1/2\end{cases}.$$

Example 4 Let U, V, A, B, B^*, and α be the same as in Example 3. Suppose that $\rightarrow=I_R$ in the α-triple I restriction method for FMT. We now calculate the α-MinT-triple I restriction solution:

$$R(u,v)=I_R(A(u),B(v))=$$

$$1-\frac{1-u}{2}+\frac{1-u}{2}\times\frac{1+v}{4}=\frac{5+3u+v-uv}{8}.$$

For (31), we have

$$(R(u,v))'+R(u,v)\times B^*(v)=1-\frac{5+3u+v-uv}{8}+$$

$$\frac{5+3u+v-uv}{8}\times\frac{1-v}{2}=\frac{11-3u-v+uv}{16}-$$

$$\frac{v(5+3u+v-uv)}{16}\leqslant\frac{12}{16}=\alpha.$$

So (31) holds (from Corollary 14(vi)). Thus it follows from Corollary 18 that the α-MinT-triple I restriction solution is

$$A^*(u)=\sup_{v\in V}\{\alpha'/[R(u,v)\times(B^*(v))']\}=$$

$$\sup_{v\in[0,1]}\{4/[(5+3u+v-uv)\times(1+v)]\},\quad u\in U.$$

Noting that $\dfrac{1}{5+3u+v-uv},\dfrac{1}{1+v}$ are decreasing w.r.t. $v(v\in V)$, we have

$$A^*(u)=\frac{4}{(5+3u)\times 1}=\frac{4}{5+3u}.$$

Remark 14 Aiming at the same U, V, A, B, B^*, and α, the α-MinT-solution in Example 3 is smaller than the

α-MinT-triple I restriction solution in Example 4 (since $1 > u \geqslant \dfrac{1}{2}$ implies $\dfrac{1}{2} < \dfrac{4}{5+3u}$, and $\dfrac{1}{2} > u \geqslant 0$ implies $\dfrac{2}{2u+3} < \dfrac{4}{5+3u}$, and $u = 1$ implies $\dfrac{1}{2} = \dfrac{4}{5+3u}$). From the α-universal triple I restriction principle for FMT (which seeks out the smallest A^* satisfying (6)), the α-universal triple I restriction method makes the reasoning closer, thus it is superior to the α-triple I restriction method.

4. Conclusions

The α-universal triple I restriction method is put forward and investigated. The main contributions and conclusions are as follows.

(i) New α-universal triple I restriction principles for FMP and FMT are brought forward, which improve the previous α-triple I restriction principles.

(ii) The α-universal triple I restriction method for FMP is investigated. The existence condition of α-FMP-solutions, and the condition that the α-SupP-quasi solution is the α-MaxP-solution, are obtained from the properties of the implication $\rightarrow_2$.

Then, aiming at the case that $\rightarrow_2$ respectively employs seven familiar implications, we achieve the corresponding expression of the α-SupP-quasi solution (or the α-MaxP-solution), as well as the necessary and sufficient condition that the α-SupP-quasi solution is the α-MaxP-solution.

(iii) The α-universal triple I restriction method for FMT is researched. The existence condition of α-FMT-solutions, and the condition that the α-InfT-quasi solution is the α-MinT-solution, are given from the properties of the implication $\rightarrow_2$.

Following that, for the case that $\rightarrow_2$ respectively takes seven implications, we obtain the corresponding expression of the α-InfT-quasi solution (or the α-MinT-solution), together with the necessary and sufficient condition (or the sufficient condition) that the α-InfT-quasi solution is the α-MinT-solution.

(iv) As a special case of the α-universal triple I restriction method, the corresponding results of the α-triple I restriction method are obtained and improved.

(v) By four examples, it is found that the α-universal triple I restriction method makes the reasoning be closer than the α-triple I restriction method, implying that the former is more reasonable (in the light of the α-universal triple I restriction principles).

In the current triple I restriction methods, there is another important form to be investigated (besides (4)), i.e.,

$$(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) < \alpha$$

where $\alpha \in (0, 1]$ (see [27,36,37]). Similar to Section 1, it is natural to research the following formula:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) < \alpha,$$

thus we get a new α-universal triple I restriction method, which will be discussed in another paper.

Moreover, we shall investigate how to construct and analyze the reasonable fuzzy systems [38,39] based on the proposed α-universal triple I restriction methods. What is more, we shall apply the proposed methods and related fuzzy systems to the fields of fuzzy control, complex system modeling and simulation, natural language processing and so on.

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