

DOUBLE FUZZY IMPLICATIONS-BASED RESTRICTION INFERENCE ALGORITHM

Y. M. TANG, X. Z. YANG, X. P. LIU AND J. YANG

ABSTRACT. The main condition of the differently implicational inference algorithm is reconsidered from a contrary direction, which motivates a new fuzzy inference strategy, called the double fuzzy implications-based restriction inference algorithm. New restriction inference principle is proposed, which improves the principle of the full implication restriction inference algorithm. Furthermore, focusing on the new algorithm, we analyze the basic property of its solution, and then obtain its optimal solutions aiming at the problems of fuzzy modus ponens (FMP) as well as fuzzy modus tollens (FMT). Lastly, comparing with the full implication restriction inference algorithm, the new algorithm can make the inference closer, and generate more, better specific inference algorithms.

1. Introduction

Two basic forms of fuzzy inference (see [2,16,22]) are fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) as following:

FMP: for a given rule $A \rightarrow B$ and input A^* , to compute B^* (output), (1)

FMT: for a given rule $A \rightarrow B$ and input B^* , to compute A^* (output), (2)

where $A, A^* \in F(U)$, $B, B^* \in F(V)$, in which $F(U), F(V)$ denote the set of all fuzzy subsets of universe U, V , respectively. To deal with these forms, the classical method is the compositional rule of inference (CRI) algorithm, proposed by Zadeh in 1973 (see [7,9,10,32]). In order to improve the CRI algorithm, Wang [29] put forward the full implication inference algorithm, which is broadly investigated by lots of scholars, see e.g. [13,15,33]. It is found that the full implication inference algorithm possesses many advantages, which are embodied as strict logic basis, reversibility and so on (see [14,18,20,24]).

The basic idea of the full implication inference algorithm is to seek out the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) such that

$$(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) \quad (3)$$

takes its maximum for any $u \in U$, $v \in V$ (where \rightarrow is a fuzzy implication on $[0, 1]$, see Definition 2.2), in which (3) is called the basic formula of the full implication inference algorithm. Moreover it is generalized to the α -full implication inference

Received: March 2013; Accepted: September 2015

Key words and phrases: Fuzzy inference, Fuzzy system, Compositional rule of inference (CRI) algorithm, Full implication inference algorithm, Fuzzy implication.

algorithm, whose solution is the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) making

$$(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) \geq \alpha \quad (4)$$

hold for any $u \in U, v \in V$ (where $\alpha \in (0, 1]$). Here (4) implies that the basic formula (3) should be greater than or equal to α , and (4) is obviously equivalent to

$$(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) \in [\alpha, 1]. \quad (5)$$

Then, what happens if the basic formula (3) belongs to the complementary set of $[\alpha, 1]$? That is,

$$(A(u) \rightarrow B(v)) \rightarrow (A^*(u) \rightarrow B^*(v)) \in [0, \alpha]. \quad (6)$$

This motivates a new, interesting strategy called the full implication restriction inference algorithm [23] (which provides necessary theoretical foundation for obtaining some performance measure of a new type of fuzzy controllers [21]), whose solution is the largest $B^* \in F(V)$ (or the smallest $A^* \in F(U)$) making (6) hold for any $u \in U, v \in V$.

On the other hand, although the full implication inference algorithm has many acknowledged advantages mentioned above, the effect of the full implication inference algorithm is imperfect (due to its inferior response ability and practicability) from the viewpoint of some kind of fuzzy controller (see [8,11,12]). To solve this problem, Tang and Liu generalized in [25] the full implication inference algorithm to the differently implicational universal triple I method of (1, 2, 2) type (the differently implicational inference algorithm for short), and then proposed the α -differently implicational inference algorithm. The α -differently implicational inference algorithm seeks the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) making

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \in [\alpha, 1] \quad (7)$$

hold for any $u \in U, v \in V$, where double fuzzy implications \rightarrow_1 and \rightarrow_2 can be different (in which $\alpha \in (0, 1]$). Moreover, it was found that the differently implicational inference algorithm can obtain more usable fuzzy controllers comparing with the CRI algorithm and full implication inference algorithm (see [25]). Later, the reversibility of differently implicational inference algorithm was discussed for FMT (where \rightarrow_2 employed I_L , and \rightarrow_1 respectively took seven familiar fuzzy implications), and obtained excellent conclusion (see [26]). In [27], the differently implicational inference algorithm was researched for FMP (where \rightarrow_2 employed I_{FD}) and applied to pattern recognition of textual emotion polarity.

For the differently implicational inference algorithm, its main condition (7) can be similarly reconsidered from a contrary direction, that is,

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \in [0, \alpha]. \quad (8)$$

The fuzzy inference strategy derived from (8) is called the double fuzzy implications-based restriction inference algorithm (DFI-restriction inference algorithm for short), which is the research aim of this paper.

2. Preliminaries

Definition 2.1. A function $T : [0, 1]^2 \rightarrow [0, 1]$ is said to be a *t-norm* if T is associative, increasing and commutative and satisfies the condition $T(1, a) = a$ ($a \in [0, 1]$).

Definition 2.2. [1, 5] A *fuzzy implication* on $[0, 1]$ is a function $I : [0, 1]^2 \rightarrow [0, 1]$ satisfying the conditions as follows:

- (C1) I is decreasing in the first variable,
- (C2) I is increasing in the second variable,
- (C3) $I(0, 0) = 1$, $I(1, 1) = 1$, $I(1, 0) = 0$.

$I(a, b)$ is also written as $a \rightarrow b$ for any $a, b \in [0, 1]$.

It is easy to find that for every fuzzy implication, $I(0, a) = I(a, 1) = 1$ ($a \in [0, 1]$) and obviously $I(0, 1) = 1$.

Definition 2.3. A function $I : [0, 1]^2 \rightarrow [0, 1]$ is said to be an *R-implication* if there exist a *t-norm* T such that

$$I(a, b) = \sup\{x \in [0, 1] \mid T(a, x) \leq b\}, \quad a, b \in [0, 1]. \quad (9)$$

Definition 2.4. Let T and I be two $[0, 1]^2 \rightarrow [0, 1]$ functions, (T, I) is called a *residual pair* or, T and I are *residual* to each other, if the following residual condition holds (iff denotes “if and only if”):

$$T(a, b) \leq c \text{ iff } b \leq I(a, c) \quad (a, b, c \in [0, 1]). \quad (10)$$

Proposition 2.5. [6] Let T be a *t-norm*, then the following statements are equivalent:

- (i) T is left-continuous;
- (ii) T and I form a residual pair, where I is obtained from (9).

Lemma 2.6. [30, 31] Suppose that T is a left-continuous *t-norm* on $[0, 1]$, and that I is an *R-implication* derived from T , then (T, I) is a residual pair, and I satisfies the following conditions:

- (C4) $a \leq I(b, c) \iff b \leq I(a, c)$,
- (C5) $I(a, I(b, c)) = I(b, I(a, c))$,
- (C6) $I(T(a, b), c) = I(a, I(b, c))$,
- (C7) $I(1, a) = a$ ($a \in [0, 1]$),
- (C8) $a \leq b \iff I(a, b) = 1$,
- (C9) $I(\sup_{x \in X} x, a) = \inf_{x \in X} I(x, a)$,
- (C10) $I(a, \inf_{x \in X} x) = \inf_{x \in X} I(a, x)$,

where $a, b, c, x \in [0, 1]$ and $X \subset [0, 1]$, $X \neq \emptyset$.

It is easy to get Proposition 2.7 from Theorem 3.1 in [25].

Proposition 2.7. Let I be a fuzzy implication satisfying

- (C11) $I(a, b)$ is right-continuous w.r.t. b ($a \in [0, 1]$, $b \in [0, 1]$),
- (C12) $\{x \in [0, 1] \mid I(a, x) = 1\} \neq \emptyset$ ($a \in [0, 1]$),

then the function $T : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$T(a, b) = \inf\{x \in [0, 1] \mid b \leq I(a, x)\}, \quad a, b \in [0, 1]$$

is residual to I , and (9) holds.

Remark 2.8. It is easy to verify that the R -implication derived from left-continuous t -norm also satisfies (C11) and (C12).

Definition 2.9. Let Z be any nonempty set and $F(Z)$ the set of all fuzzy subsets on Z , define partial order relation \leq_F on $F(Z)$ (according to pointwise order) as: $A \leq_F B$ iff $A(z_0) \leq B(z_0)$ for any $z_0 \in Z$, where $A, B \in F(Z)$.

Lemma 2.10. $\langle F(Z), \leq_F \rangle$ is a complete lattice.

3. The DFI-restriction Inference Algorithm for FMP

For the FMP problem (1), from the viewpoint of the double fuzzy implications-based restriction inference algorithm, we can obtain the following restriction inference principle:

DFI-restriction Inference Principle for FMP: The conclusion B^* of FMP problem (1) is the largest fuzzy set satisfying (8) in $\langle F(V), \leq_F \rangle$.

It is obvious that such DFI-restriction inference principle for FMP improves the full implication restriction inference principle w.r.t. (6) for FMP in [23].

Definition 3.1. Let $A, A^* \in F(U)$, $B \in F(V)$, if B^* (in $\langle F(V), \leq_F \rangle$) makes (8) hold for any $u \in U, v \in V$, then B^* is called a DFI-restriction inference solution of FMP (DFI-solution of FMP for short).

Theorem 3.2. Assume that $A, A^* \in F(U)$, $B \in F(V)$, $\alpha \in (0, 1]$. Then there exists a $B^* \in F(V)$ as a DFI-solution of FMP iff the following inequality holds for any $u \in U, v \in V$:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 0) < \alpha. \quad (11)$$

Proof. On the one hand, if (11) holds, then we take $B^*(v) \equiv 0$, thus B^* obviously satisfies (8), and hence B^* is a DFI-solution of FMP. On the other hand, if there exists a $B^* \in F(V)$ which is a DFI-solution of FMP, then B^* satisfies (8), and because \rightarrow_2 satisfies (C2) we have that $A^*(u) \rightarrow_2 B^*(v) \geq A^*(u) \rightarrow_2 0$, and $\alpha > (A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \geq (A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 0)$, i.e. (11) holds. \square

Similar to Theorem 3.2, we can prove Proposition 3.3.

Proposition 3.3. Suppose that D_1 is a DFI-solution of FMP, and that $D_2 \leq_F D_1$ (in which $D_1, D_2 \in \langle F(V), \leq_F \rangle$). Then D_2 is a DFI-solution of FMP.

Remark 3.4. Suppose that (11) holds. For a DFI-solution of FMP D_1^* , every fuzzy set D_2^* which is less than D_1^* , will be a DFI-solution of FMP (see Proposition 3.3, where $D_1^*, D_2^* \in \langle F(V), \leq_F \rangle$). This means that there are many DFI-solutions of FMP, including $D_3^*(v) \equiv 0$ ($v \in V$). The last D_3^* is a special solution, for which (8) always holds no matter what major premise $A \rightarrow_1 B$ and minor premise A^* are adopted. Therefore, when the optimal DFI-solution of FMP exists, it should be the largest one; in other words, it should be the supremum of all solutions.

Theorem 3.5. *If \rightarrow_2 is a fuzzy implication satisfying (C11) and (C12), and T is the function residual to \rightarrow_2 , and $A, A^* \in F(U)$, $B \in F(V)$, $\alpha \in (0, 1]$, (11) holds, then the supremum of DFI-solutions of FMP can be computed as follows:*

$$B^*(v) = \inf_{u \in U} T(A^*(u), T(A(u) \rightarrow_1 B(v), \alpha)), v \in V. \quad (12)$$

Proof. Since \rightarrow_2 is a fuzzy implication satisfying (C11) and (C12), it follows from Proposition 2.7 that the residual condition (10) holds.

Let $G_1 = \{v \in V \mid B^*(v) = 0\}$, and $G_2 = \{v \in V \mid B^*(v) > 0\}$. Suppose that $C \in F(V)$, and that $C(v) = 0$ for $v \in G_1$, and that $C(v) < B^*(v)$ for $v \in G_2$. We shall show that C is a DFI-solution of FMP, that is, the following inequality holds for any $u \in U, v \in V$:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 C(v)) < \alpha. \quad (13)$$

If $v \in G_1$, then it follows from (11) that $C(v) = 0$ satisfies (13) for any $u \in U$.

If $v \in G_2$, then it follows from (12) and $C(v) < B^*(v)$ that

$$C(v) < T(A^*(u), T(A(u) \rightarrow_1 B(v), \alpha)) \quad (14)$$

holds for any $u \in U$. Suppose, on the contrary, that (13) does not hold. Then there exist $u_0 \in U$ and $v_0 \in V$ such that

$$(A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (A^*(u_0) \rightarrow_2 C(v_0)) \geq \alpha$$

holds (obviously $v_0 \in G_2$). Thus it follows from the residual condition (10) that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq A^*(u_0) \rightarrow_2 C(v_0)$, and $T(A^*(u_0), T(A(u_0) \rightarrow_1 B(v_0), \alpha)) \leq C(v_0)$, which contradicts (14). Thus (13) holds for any $u \in U, v \in V$. As a result, C is a DFI-solution of FMP.

Next, we shall check that B^* determined by (12) is the supremum of DFI-solutions of FMP. Assume that $D \in F(V)$, and that there exists $v_0 \in V$ such that $D(v_0) > B^*(v_0)$. We shall prove that D is not a DFI-solution of FMP. In fact, it follows from (12) that there exists $u_0 \in U$ such that

$$D(v_0) > T(A^*(u_0), T(A(u_0) \rightarrow_1 B(v_0), \alpha))$$

holds. We have from the residual condition (10) that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq A^*(u_0) \rightarrow_2 D(v_0)$, and $\alpha \leq (A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (A^*(u_0) \rightarrow_2 D(v_0))$. Thus, D is not a DFI-solution of FMP.

To sum up, B^* determined by (12) is the supremum of DFI-solutions of FMP. \square

It follows from Theorem 3.5 and Remark 2.8 that we have Corollary 3.6.

Corollary 3.6. *If \rightarrow_2 is an R -implication derived from left-continuous t -norm T , and $A, A^* \in F(U)$, $B \in F(V)$, $\alpha \in (0, 1]$, (11) holds, then the supremum of DFI-solutions of FMP is as follows:*

$$B^*(v) = \inf_{u \in U} T(A^*(u), T(A(u) \rightarrow_1 B(v), \alpha)), v \in V.$$

Example 3.7. *The following functions are fuzzy implications satisfying (C11) and (C12), where I_{FD} is from [4] (which is also called I_0 implication, see [17,29]), and I_{EP}, I_{Y_α} are from [25,28], and I_{RR} is from [3] (in which $a, b \in [0, 1]$, and x' denotes $1 - x$).*

$$\begin{aligned}
I_L(a, b) &= \begin{cases} 1, & a \leq b \\ a' + b, & a > b \end{cases} \text{ (Lukasiewicz implication);} \\
I_G(a, b) &= \begin{cases} 1, & a \leq b \\ b, & a > b \end{cases} \text{ (Gödel implication);} \\
I_{Go}(a, b) &= \begin{cases} 1, & a = 0 \\ (b/a) \wedge 1, & a \neq 0 \end{cases} \text{ (Goguen implication);} \\
I_{FD}(a, b) &= \begin{cases} 1, & a \leq b \\ a' \vee b, & a > b \end{cases} \text{ (Fodor implication);} \\
I_{GR}(a, b) &= \begin{cases} 1, & a \leq b \\ 0, & a > b \end{cases} \text{ (Gaines-Rescher implication);} \\
I_Y(a, b) &= b^a \text{ (} I_Y(0, 0) = 1 \text{) (Yager implication);} \\
I_R(a, b) &= a' + a \times b \text{ (Reichenbach implication);} \\
I_{Y_a}(a, b) &= \begin{cases} 1, & a \leq b \\ 1 - (\sqrt{1-b} - \sqrt{1-a})^2, & a > b \end{cases}; \\
I_{EP}(a, b) &= \begin{cases} 1, & a \leq b \\ (2b - ab)/(a + b - ab), & a > b \end{cases}; \\
I_{RR}(a, b) &= \begin{cases} 1, & a \leq b \\ 1 - a + ab, & a > b \end{cases} \text{ (revised Reichenbach implication).}
\end{aligned}$$

Moreover, the functions respectively residual to $I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Y_a}, I_{EP}, I_{RR}$ are as follows.

$$\begin{aligned}
T_L(a, b) &= \begin{cases} a + b - 1, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases}, \\
T_G(a, b) &= a \wedge b, \\
T_{Go}(a, b) &= a \times b, \\
T_{FD}(a, b) &= \begin{cases} a \wedge b, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases}, \\
T_{GR}(a, b) &= \begin{cases} a, & b > 0 \\ 0, & b = 0 \end{cases}, \\
T_Y(a, b) &= \begin{cases} \sqrt[a]{b}, & a > 0 \\ 0, & a = 0 \end{cases}, \\
T_R(a, b) &= \begin{cases} (a + b - 1)/a, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases}, \\
T_{Y_a}(a, b) &= \begin{cases} 1 - (g(a, b))^2, & g(a, b) \leq 1 \\ 0, & g(a, b) > 1 \end{cases} \text{ where } g(a, b) = \sqrt{1-a} + \sqrt{1-b}, \\
T_{EP}(a, b) &= ab/[2 - (a + b - ab)], \\
T_{RR}(a, b) &= \begin{cases} [(a + b - 1)/a] \wedge a, & a + b > 1 \\ 0, & a + b \leq 1 \end{cases}.
\end{aligned}$$

Here T_{EP} is the Einstein product; and T_{Y_a} is the t -norm of Yager (where ω takes 0.5), which is defined as

$$T_{Y_{a-\omega}}(a, b) = 1 - \min[1, ((1-a)^\omega + (1-b)^\omega)^{1/\omega}], \text{ in which } \omega \in (0, \infty).$$

It is easy to know that $T_L, T_G, T_{Go}, T_{FD}, T_{EP}, T_{Y_a}$ are left-continuous t -norm, thus $I_L, I_G, I_{Go}, I_{FD}, I_{y-0.5}, I_{EP}$ are R -implications derived from left-continuous t -norm.

Proposition 3.8. *Suppose that $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$ in (8), and that (11) holds, then the supremum of DFI-solutions of FMP is as follows, respectively ($v \in V$):*

- (i) *If \rightarrow_2 takes I_L , then $B^*(v) = \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 2\}$;*
- (ii) *If \rightarrow_2 takes I_G , then $B^*(v) = \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\} \wedge \alpha$;*
- (iii) *If \rightarrow_2 takes I_{Go} , then $B^*(v) = \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha\}$;*
- (iv) *If \rightarrow_2 takes I_{FD} , then $B^*(v) = \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\} \wedge \alpha$;*
- (v) *If \rightarrow_2 takes I_{GR} , then $B^*(v) = \inf_{u \in U} \{A^*(u)\}$;*
- (vi) *If \rightarrow_2 takes I_Y , then $B^*(v) = \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \sqrt{\alpha}\}$;*
- (vii) *If \rightarrow_2 takes I_R , then $B^*(v) = \inf_{u \in U} \{[A^*(u) + \varphi_R(u, v) - 1]/A^*(u)\}$ where $\varphi_R(u, v) = [(A(u) \rightarrow_1 B(v)) + \alpha - 1]/(A(u) \rightarrow_1 B(v))$;*
- (viii) *If \rightarrow_2 takes I_{Ya} , then $B^*(v) = \inf_{u \in U} \{1 - (\sqrt{1 - \alpha} + \sqrt{1 - A^*(u)} + \sqrt{1 - (A(u) \rightarrow_1 B(v))})^2\}$;*
- (ix) *If \rightarrow_2 takes I_{EP} , then $B^*(v) = \inf_{u \in U} \{A^*(u) \times \varphi_{EP}(u, v)/[2 - A^*(u) - \varphi_{EP}(u, v) + A^*(u) \times \varphi_{EP}(u, v)]\}$ where $\varphi_{EP}(u, v) = \alpha \times (A(u) \rightarrow_1 B(v))/[2 - (A(u) \rightarrow_1 B(v)) - \alpha + \alpha \times (A(u) \rightarrow_1 B(v))]$;*
- (x) *If \rightarrow_2 takes I_{RR} , then $B^*(v) = \inf_{u \in U} \{[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u)\}$ where $\varphi_{RR}(u, v) = [(A(u) \rightarrow_1 B(v)) + \alpha - 1]/(A(u) \rightarrow_1 B(v)) \wedge (A(u) \rightarrow_1 B(v))$.*

Proof. If $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, then \rightarrow_2 is a fuzzy implication satisfying (C11) and (C12), thus it follows from Theorem 3.5 that the supremum of DFI-solutions of FMP is $B^*(v) = \inf_{u \in U} T(A^*(u), T(A(u) \rightarrow_1 B(v), \alpha))$, $v \in V$ (where T is the function residual to \rightarrow_2). Then, we need to get the specific expression of B^* . We only prove the cases of I_L, I_{RR} as examples, the remainders can be proved similarly.

(i) Suppose that \rightarrow_2 takes I_L . It follows from Example 3.7 that T_L is the function residual to I_L . By (11), we have that $A^*(u) > 0$, $A(u) \rightarrow_1 B(v) > 1 - A^*(u)$, and $1 - (A(u) \rightarrow_1 B(v)) + 1 - A^*(u) < \alpha$ hold for any $u \in U, v \in V$. We further get that $(A(u) \rightarrow_1 B(v)) + \alpha > 1$ and $A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 1 > 1$ hold for any $u \in U, v \in V$. Thus,

$$\begin{aligned} B^*(v) &= \inf_{u \in U} T_L(A^*(u), T_L(A(u) \rightarrow_1 B(v), \alpha)) \\ &= \inf_{u \in U} T_L(A^*(u), ((A(u) \rightarrow_1 B(v)) + \alpha - 1)) \\ &= \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 2\}, v \in V. \end{aligned}$$

(xi) Suppose that \rightarrow_2 takes I_{RR} . We get from Example 3.7 that T_{RR} is the function residual to I_{RR} . It follows from (11) that $A^*(u) > 0$, $A(u) \rightarrow_1 B(v) > 1 - A^*(u)$, and $1 - (A(u) \rightarrow_1 B(v)) + (A(u) \rightarrow_1 B(v)) \times (1 - A^*(u)) < \alpha$ hold for any $u \in U, v \in V$. We further have

$$\begin{aligned} (A(u) \rightarrow_1 B(v)) + A^*(u) &> 1, \\ (A(u) \rightarrow_1 B(v)) + \alpha &> 1, \\ 1 - A^*(u) \times (A(u) \rightarrow_1 B(v)) &< \alpha. \end{aligned}$$

Then $A^*(u) + \{[(A(u) \rightarrow_1 B(v)) + \alpha - 1]/(A(u) \rightarrow_1 B(v))\} \wedge (A(u) \rightarrow_1 B(v)) > 1$ holds for any $u \in U, v \in V$. Summarizing above, we obtain that

$$\begin{aligned} B^*(v) &= \inf_{u \in U} T_{RR}(A^*(u), T_{RR}(A(u) \rightarrow_1 B(v), \alpha)) \\ &= \inf_{u \in U} T_{RR}(A^*(u), \varphi_{RR}(u, v)) \\ &= \inf_{u \in U} \{[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u)\}, v \in V \end{aligned}$$

where $\varphi_{RR}(u, v) \triangleq [(A(u) \rightarrow_1 B(v)) + \alpha - 1]/(A(u) \rightarrow_1 B(v)) \wedge (A(u) \rightarrow_1 B(v))$. \square

Theorem 3.9. *If $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, $A, A^* \in F(U)$, $B \in F(V)$, $\alpha \in (0, 1]$, (11) holds, then the condition, which the supremum of DFI-solutions of FMP is the maximum, is as follows, respectively (for any $u \in U, v \in V$):*

(i) *Let \rightarrow_2 take I_L , $A^*(u) + (A(u) \rightarrow_1 B(v)) > \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v))\}$ holds;*

(ii) *Let \rightarrow_2 take I_G , $(A^*(u) \wedge (A(u) \rightarrow_1 B(v))) \wedge \alpha > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\}$ holds;*

(iii) *Let \rightarrow_2 take I_{Go} , $A^*(u) \times (A(u) \rightarrow_1 B(v)) > \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v))\}$ holds;*

(iv) *Let \rightarrow_2 take I_{FD} , $(A^*(u) \wedge (A(u) \rightarrow_1 B(v))) \wedge \alpha > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\}$ holds;*

(v) *Let \rightarrow_2 take I_{GR} , $A^*(u) > \inf_{u \in U} \{A^*(u)\}$ holds;*

(vi) *Let \rightarrow_2 take I_Y , $A^*(u) \times (A(u) \rightarrow_1 B(v)) \sqrt{\alpha} > \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \sqrt{\alpha}\}$ holds;*

(vii) *Let \rightarrow_2 take I_R , $[A^*(u) + \varphi_R(u, v) - 1]/A^*(u) > \inf_{u \in U} \{[A^*(u) + \varphi_R(u, v) - 1]/A^*(u)\}$ holds;*

(viii) *Let \rightarrow_2 take I_{Ya} , $1 - (\sqrt{1 - A^*(u)} + \sqrt{1 - (A(u) \rightarrow_1 B(v))} + \sqrt{1 - \alpha})^2 > \inf_{u \in U} \{1 - (\sqrt{1 - A^*(u)} + \sqrt{1 - (A(u) \rightarrow_1 B(v))} + \sqrt{1 - \alpha})^2\}$ holds;*

(ix) *Let \rightarrow_2 take I_{EP} , $A^*(u) \times \varphi_{EP}(u, v)/[2 - A^*(u) - \varphi_{EP}(u, v) + A^*(u) \times \varphi_{EP}(u, v)] > \inf_{u \in U} \{A^*(u) \times \varphi_{EP}(u, v)/[2 - A^*(u) - \varphi_{EP}(u, v) + A^*(u) \times \varphi_{EP}(u, v)]\}$ holds;*

(x) *Let \rightarrow_2 take I_{RR} , $[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u) > \inf_{u \in U} \{[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u)\}$ holds.*

Proof. Note that if the supremum B^* of DFI-solutions of FMP is a DFI-solution of FMP, then B^* is the maximum of DFI-solutions of FMP. Thus it is enough to prove that B^* is a DFI-solution of FMP (i.e., B^* should make (8) hold for any $u \in U, v \in V$). We still prove the cases of I_L, I_{RR} as examples.

(i) Let \rightarrow_2 take I_L . It follows from Proposition 3.8 that the supremum of DFI-solutions of FMP is $B^*(v) = \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 2\}$. By the condition given in (i), we obtain

$$\begin{aligned} B^*(v) &= \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 2\} \\ &= \alpha - 2 + \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v))\} \\ &< \alpha - 2 + A^*(u) + (A(u) \rightarrow_1 B(v)). \end{aligned}$$

Note that $0 \geq \alpha - 2 + (A(u) \rightarrow_1 B(v))$, then $A^*(u) \geq A^*(u) + \alpha - 2 + (A(u) \rightarrow_1 B(v)) > B^*(v)$ holds. Thus

$$\begin{aligned} A(u) \rightarrow_1 B(v) &\geq \alpha - 1 + (A(u) \rightarrow_1 B(v)) \\ &= (1 - A^*(u)) + [\alpha - 2 + A^*(u) + (A(u) \rightarrow_1 B(v))] \\ &> 1 - A^*(u) + B^*(v). \end{aligned}$$

To sum up, we get $(u \in U, v \in V)$

$$\begin{aligned} &(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \\ &= (A(u) \rightarrow_1 B(v)) \rightarrow_2 [1 - A^*(u) + B^*(v)] \\ &= 1 - (A(u) \rightarrow_1 B(v)) + 1 - A^*(u) + B^*(v) \\ &< 1 - (A(u) \rightarrow_1 B(v)) + 1 - A^*(u) + \alpha - 2 + A^*(u) + (A(u) \rightarrow_1 B(v)) \\ &= \alpha. \end{aligned}$$

Thus B^* makes (8) hold for any $u \in U, v \in V$, and hence it is a DFI-solution of FMP.

(x) Let \rightarrow_2 take I_{RR} . It follows from Proposition 3.8 that the supremum of DFI-solutions of FMP is as follows:

$$B^*(v) = \inf_{u \in U} \{[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u)\}.$$

For convenience, denote $R_1(u, v) = A(u) \rightarrow_1 B(v)$. It follows from (11) that $A^*(u) > 0$, $R_1(u, v) > 1 - A^*(u)$, and $1 - R_1(u, v) \times A^*(u) < \alpha$ hold for any $u \in U, v \in V$. By the condition given in (x), we obtain

$$\begin{aligned} B^*(v) &< [(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u) \\ &= \frac{A^*(u) + [((R_1(u, v) + \alpha - 1)/R_1(u, v)) \wedge R_1(u, v)] - 1}{A^*(u)} \wedge A^*(u). \end{aligned}$$

Further, we get $A^*(u) > B^*(v)$, and

$$\begin{aligned} B^*(v) &< \frac{A^*(u) + [(R_1(u, v) + \alpha - 1)/R_1(u, v)] - 1}{A^*(u)} \\ &= \frac{R_1(u, v) \times A^*(u) + \alpha - 1}{R_1(u, v) \times A^*(u)}, \\ B^*(v) &< (A^*(u) + R_1(u, v) - 1)/A^*(u), \end{aligned}$$

i.e., $R_1(u, v) > 1 - A^*(u) + A^*(u) \times B^*(v)$.

To sum up, we achieve $(u \in U, v \in V)$

$$\begin{aligned} &(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \\ &= R_1(u, v) \rightarrow_2 [1 - A^*(u) + A^*(u)B^*(v)] \\ &= 1 - R_1(u, v) + R_1(u, v) \times [1 - A^*(u) + A^*(u)B^*(v)] \\ &= 1 - R_1(u, v) \times A^*(u) + R_1(u, v) \times A^*(u) \times B^*(v) \\ &< 1 - R_1(u, v) \times A^*(u) + R_1(u, v) \times A^*(u) \times \\ &\quad [(R_1(u, v) \times A^*(u) + \alpha - 1)/(R_1(u, v) \times A^*(u))] \\ &= 1 - R_1(u, v) \times A^*(u) + R_1(u, v) \times A^*(u) + \alpha - 1 \\ &= \alpha. \end{aligned}$$

Therefore B^* makes (8) hold for any $u \in U, v \in V$, and hence it is a DFI-solution of FMP. \square

\rightarrow_2	The supremum B^*	The condition which B^* is the maximum
I_L	$\inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v)) + \alpha - 2\}$	$A^*(u) + (A(u) \rightarrow_1 B(v)) > \inf_{u \in U} \{A^*(u) + (A(u) \rightarrow_1 B(v))\}$
I_G	$\inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\} \wedge \alpha$	$(A^*(u) \wedge (A(u) \rightarrow_1 B(v))) \wedge \alpha > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\}$
I_{Go}	$\inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha\}$	$A^*(u) \times (A(u) \rightarrow_1 B(v)) > \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v))\}$
I_{FD}	$\inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\} \wedge \alpha$	$(A^*(u) \wedge (A(u) \rightarrow_1 B(v))) \wedge \alpha > \inf_{u \in U} \{A^*(u) \wedge (A(u) \rightarrow_1 B(v))\}$
I_{GR}	$\inf_{u \in U} \{A^*(u)\}$	$A^*(u) > B^*$
I_Y	$\inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \sqrt{\alpha}\}$	$A^*(u) \times (A(u) \rightarrow_1 B(v)) \sqrt{\alpha} > B^*$
I_R	$\inf_{u \in U} \{[A^*(u) + \varphi_R(u, v) - 1]/A^*(u)\}$	$[A^*(u) + \varphi_R(u, v) - 1]/A^*(u) > B^*$
I_{Ya}	$\inf_{u \in U} \{1 - (\sqrt{1 - A^*(u)} + \sqrt{1 - (A(u) \rightarrow_1 B(v))})^2\}$	$1 - (\sqrt{1 - A^*(u)} + \sqrt{1 - (A(u) \rightarrow_1 B(v))})^2 > B^*$
I_{EP}	$\inf_{u \in U} \{A^*(u) \times \varphi_{EP}(u, v)/[2 - A^*(u) - \varphi_{EP}(u, v) + A^*(u) \times \varphi_{EP}(u, v)]\}$	$A^*(u) \times \varphi_{EP}(u, v)/[2 - A^*(u) - \varphi_{EP}(u, v) + A^*(u) \times \varphi_{EP}(u, v)] > B^*$
I_{RR}	$\inf_{u \in U} \{[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u)\}$	$[(A^*(u) + \varphi_{RR}(u, v) - 1)/A^*(u)] \wedge A^*(u) > B^*$

TABLE 1. Some Conclusions of the DFI-restriction Inference Algorithm for FMP

In the following Table 1 we show some conclusions of the DFI-restriction inference algorithm for FMP from Proposition 3.8 and Theorem 3.9.

Example 3.10. Let $U = V = [0, 1]$, $A(u) = (2 + u)/4$, $B(v) = (1 + 2v)/4$, $A^*(u) = (2 - u)/2$, $\alpha = 1/2$, in which $u \in U, v \in V$. Assume that $\rightarrow_2 = I_{Go}$, $\rightarrow_1 = I_L$ in the DFI-restriction inference algorithm for FMP. We now compute the supremum of DFI-solutions of FMP.

$$\begin{aligned}
A(u) \rightarrow_1 B(v) &= I_L(A(u), B(v)) = \begin{cases} 1 - \frac{2+u}{4} + \frac{1+2v}{4}, & \text{if } \frac{2+u}{4} > \frac{1+2v}{4} \\ 1, & \text{if } \frac{2+u}{4} \leq \frac{1+2v}{4} \end{cases} \\
&= \begin{cases} \frac{3-u+2v}{4}, & \text{if } 1+u > 2v \\ 1, & \text{if } 1+u \leq 2v \end{cases}.
\end{aligned}$$

Here (11) evidently holds. Then it follows from Proposition 3.8 (iii) that the supremum of DFI-solutions of FMP is as follows ($v \in V$):

$$\begin{aligned}
B^*(v) &= \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha\} \\
&= \inf_{u \in [0,1]} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha \mid 1+u > 2v\} \\
&\quad \wedge \inf_{u \in [0,1]} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha \mid 1+u \leq 2v\} \\
&= \inf_{u \in [0,1]} \left\{ \frac{2-u}{4} \times \frac{3-u+2v}{4} \mid 1+u > 2v \right\} \wedge \inf_{u \in [0,1]} \left\{ \frac{2-u}{4} \mid 1+u \leq 2v \right\}.
\end{aligned}$$

(i) Suppose $v = 1$, then $\{u \in [0, 1] \mid 1 + u > 2v\} = \emptyset$, and $1 \in \{u \in [0, 1] \mid 1 + u \leq 2v\}$. Note that $\frac{2-u}{4}$ is decreasing w.r.t. u , thus we obtain $B^*(v) = (\inf \emptyset) \wedge \frac{1}{4} = 1 \wedge \frac{1}{4} = \frac{1}{4} = \frac{1+v}{8}$.

(ii) Suppose $1 > v \geq 1/2$, then $1 \in \{u \in [0, 1] \mid 1 + u > 2v\}$. Since $\frac{2-u}{4}$, $\frac{3-u+2v}{4}$ are decreasing w.r.t. u , we get $B^*(v) = \frac{2+2v}{16} \wedge \frac{2-(2v-1)}{4} = \frac{1+v}{8} \wedge \frac{3-2v}{4} = \frac{1+v}{8}$, where $\frac{1+v}{8} < \frac{3-2v}{4}$ since $1 > v \geq 1/2$.

(iii) Suppose $0 \leq v < 1/2$, then $1 \in \{u \in [0, 1] \mid 1 + u > 2v\}$, and $\{u \in [0, 1] \mid 1 + u \leq 2v\} = \emptyset$, thus we have $B^*(v) = \frac{1+v}{8} \wedge (\inf \emptyset) = \frac{1+v}{8}$. Together we achieve $B^*(v) = \frac{1+v}{8}$, $v \in V$.

Example 3.11. Let U, V, A, B, A^*, α be the same as in Example 3.10. Assume that $\rightarrow_2 = I_{Go}$, $\rightarrow_1 = I_{Go}$ in the DFI-restriction inference algorithm for FMP, which degenerates into the full implication restriction inference algorithm for FMP (taking I_{Go}). We now compute the supremum of DFI-solutions of FMP.

$$\begin{aligned} A(u) \rightarrow_1 B(v) &= I_{Go}(A(u), B(v)) = \begin{cases} \frac{1+2v}{2+u}, & \text{if } \frac{2+u}{4} > \frac{1+2v}{4} \\ 1, & \text{if } \frac{2+u}{4} \leq \frac{1+2v}{4} \end{cases} \\ &= \begin{cases} \frac{1+2v}{2+u}, & \text{if } 1+u > 2v \\ 1, & \text{if } 1+u \leq 2v \end{cases}. \end{aligned}$$

Here (11) obviously holds. Then we get from Proposition 3.8 (iii) that the supremum of DFI-solutions of FMP is as follows ($v \in V$):

$$\begin{aligned} B^*(v) &= \inf_{u \in U} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha\} \\ &= \inf_{u \in [0, 1]} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha \mid 1 + u > 2v\} \\ &\quad \wedge \inf_{u \in [0, 1]} \{A^*(u) \times (A(u) \rightarrow_1 B(v)) \times \alpha \mid 1 + u \leq 2v\} \\ &= \inf_{u \in [0, 1]} \left\{ \frac{2-u}{4} \times \frac{1+2v}{2+u} \mid 1+u > 2v \right\} \wedge \inf_{u \in [0, 1]} \left\{ \frac{2-u}{4} \mid 1+u \leq 2v \right\} \\ &= \inf_{u \in [0, 1]} \left\{ \frac{(2-u)(1+2v)}{4(2+u)} \mid 1+u > 2v \right\} \wedge \inf_{u \in [0, 1]} \left\{ \frac{2-u}{4} \mid 1+u \leq 2v \right\}. \end{aligned}$$

(i) Suppose $v = 1$, then $\{u \in [0, 1] \mid 1 + u > 2v\} = \emptyset$, and $1 \in \{u \in [0, 1] \mid 1 + u \leq 2v\}$. Since $\frac{2-u}{4}$ is decreasing w.r.t. u , we have

$$B^*(v) = (\inf \emptyset) \wedge \frac{1}{4} = 1 \wedge \frac{1}{4} = \frac{1}{4} = \frac{1+2v}{12}.$$

(ii) Suppose $1 > v \geq 1/2$, then $1 \in \{u \in [0, 1] \mid 1 + u > 2v\}$. Noting that $\frac{2-u}{4}$, $\frac{(2-u)(1+2v)}{4(2+u)}$ are decreasing w.r.t. u , we get

$$B^*(v) = \frac{1+2v}{12} \wedge \frac{2-(2v-1)}{4} = \frac{1+2v}{12} \wedge \frac{3-2v}{4} = \frac{1+2v}{12},$$

where $\frac{1+2v}{12} < \frac{3-2v}{4}$ since $1 > v \geq 1/2$.

(iii) Suppose $0 \leq v < 1/2$, then $1 \in \{u \in [0, 1] \mid 1 + u > 2v\}$, and $\{u \in [0, 1] \mid 1 + u \leq 2v\} = \emptyset$, thus we have

$$B^*(v) = \frac{1+2v}{12} \wedge (\inf \emptyset) = \frac{1+2v}{12} \wedge 1 = \frac{1+2v}{12}.$$

	DFI-restriction inference algorithm for FMP	Full implication restriction inference algorithm for FMP
Specific ones	100 kinds	10 kinds
Actual specific ones	81 kinds	9 kinds

TABLE 2. Specific Fuzzy Inference Algorithms for FMP

Together we obtain $B^*(v) = \frac{1+2v}{12}$, $v \in V$.

Remark 3.12. For the same U, V, A, B, A^*, α in Example 3.10 and Example 3.11, because

$$\begin{aligned} 0 \leq v < 1 &\Rightarrow \frac{1+v}{8} > \frac{1+2v}{12}, \\ v = 1 &\Rightarrow \frac{1+v}{8} = \frac{1+2v}{12}, \end{aligned}$$

the supremum of DFI-solutions of FMP (from the DFI-restriction inference algorithm) in Example 3.10 is larger than the one (from the full implication restriction inference algorithm) in Example 3.11. From the viewpoint of DFI-restriction inference principle for FMP (which seeks the largest B^* satisfying (8)), the DFI-restriction inference algorithm for FMP in Examples 3.10 lets the inference closer, then it is more reasonable than the full implication restriction inference algorithm for FMP in Examples 3.11.

Remark 3.13. For the set of fuzzy implications $\{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Y\alpha}, I_{EP}, I_{RR}\}$, it follows from the DFI-restriction inference algorithm that we can get

$$10 * 10 = 100$$

kinds of specific fuzzy inference algorithms for FMP, where $\rightarrow_2, \rightarrow_1$ respectively employ these 10 fuzzy implications. Furthermore, there are

$$8 * 10 + 1 = 81$$

kinds of actual specific algorithms for FMP (derived from the DFI-restriction inference algorithm), because the expression is independent of \rightarrow_1 for the case $\rightarrow_2 = I_{GR}$, and the expressions are the same for the cases $\rightarrow_2 = I_G$ and $\rightarrow_2 = I_{FD}$ (according to Table 1). But, from the full implication restriction inference algorithm, it is easy to find that there are only 10 kinds of specific algorithms for FMP (including 9 kinds of actual specific algorithms for FMP), which is shown as Table 2. As a result, the DFI-restriction inference algorithm can provide more and better specific algorithms for FMP (than the full implication restriction inference algorithm), so it is better than the full implication restriction inference algorithm.

4. The DFI-restriction Inference Algorithm for FMT

For the FMT problem (2), aiming at the double fuzzy implications-based restriction inference algorithm, we can get the following restriction inference principle (which similarly improves the full implication restriction inference principle w.r.t. (6) for FMT in [23]):

DFI-restriction Inference Principle for FMT: The conclusion A^* of FMT problem (2) is the smallest fuzzy set satisfying (8) in $\langle F(U), \leq_F \rangle$.

Definition 4.1. Let $A \in F(U)$, $B, B^* \in F(V)$, if A^* (in $\langle F(U), \leq_F \rangle$) makes (8) hold for any $u \in U$ and $v \in V$, then A^* is called a DFI-restriction inference solution of FMT (DFI-solution of FMT for short).

Similar to Theorem 3.2 and Proposition 3.3, we can get Theorem 4.2 and Proposition 4.3.

Theorem 4.2. Assume that $A \in F(U)$, $B, B^* \in F(V)$, $\alpha \in (0, 1]$. Then there exists an $A^* \in F(U)$ as a DFI-solution of FMT iff the following inequality holds for any $u \in U, v \in V$:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (1 \rightarrow_2 B^*(v)) < \alpha. \quad (15)$$

Proposition 4.3. Suppose that C_1 is a DFI-solution of FMT, and that $C_1 \leq_F C_2$ (in which $C_1, C_2 \in \langle F(U), \leq_F \rangle$). Then C_2 is a DFI-solution of FMT.

Remark 4.4. Suppose that (15) holds. For a DFI-solution of FMT C_1^* , every fuzzy set C_2^* which is larger than C_1^* , will be a DFI-solution of FMT (see Proposition 4.3, where $C_1^*, C_2^* \in \langle F(U), \leq_F \rangle$). This implies that there are many DFI-solutions of FMT, including $C_3^*(u) \equiv 1$ ($u \in U$). The last C_3^* is a special solution, since (8) always holds no matter what $A \rightarrow_1 B$ and B^* are adopted. As a result, when the optimal DFI-solution of FMT exists, it should be the smallest one; in other words, it should be the infimum of all solutions.

Example 4.5. From Theorem 4.2, we know that if $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_Y, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, then there exists an $A^* \in F(U)$ which is a DFI-solution of FMT iff (15) holds for any $u \in U, v \in V$.

Theorem 4.6. If \rightarrow_2 is a fuzzy implication satisfying (C4), (C11) and (C12), and T is the function residual to \rightarrow_2 , and $A \in F(U)$, $B, B^* \in F(V)$, $\alpha \in (0, 1]$, (15) holds, then the infimum of DFI-solutions of FMT can be computed as follows:

$$A^*(u) = \sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\}, u \in U. \quad (16)$$

Proof. Since the fuzzy implication \rightarrow_2 satisfies (C11) and (C12), it follows from Proposition 2.7 that the residual condition (10) holds.

Let $H_1 = \{u \in U \mid A^*(u) = 1\}$ and $H_2 = \{u \in U \mid A^*(u) < 1\}$. Assume that $C \in F(U)$, and that $C(u) = 1$ for $u \in H_1$, and that $C(u) > A^*(u)$ for $u \in H_2$. We shall show that C is a DFI-solution of FMT, that is, the following inequality holds for any $u \in U, v \in V$:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (C(u) \rightarrow_2 B^*(v)) < \alpha. \quad (17)$$

If $u \in H_1$, then it follows from (15) that $C(u) = 1$ satisfies (17) for any $v \in V$.

If $u \in H_2$, then it follows from (16) and $C(u) > A^*(u)$ that

$$C(u) > T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v) \quad (18)$$

holds for any $v \in V$. Suppose, on the contrary, that (17) does not hold. Then there exist $u_0 \in U$ and $v_0 \in V$ such that $(A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (C(u_0) \rightarrow_2 B^*(v_0)) \geq \alpha$ holds (obviously $u_0 \in H_2$). Thus it follows from residual condition (10) that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq C(u_0) \rightarrow_2 B^*(v_0)$ holds, and considering that \rightarrow_2 satisfies (C4), we have $C(u_0) \leq T(A(u_0) \rightarrow_1 B(v_0), \alpha) \rightarrow_2 B^*(v_0)$, which contradicts (18). Therefore (17) holds for any $u \in U, v \in V$. As a result, C is a DFI-solution of FMT.

Next, we shall check that A^* determined by (16) is the infimum of DFI-solutions of FMT. Assume that $D \in F(U)$, and that there exists $u_0 \in U$ such that $D(u_0) < A^*(u_0)$. We shall prove that D is not a DFI-solution of FMT. In fact, it follows from (16) that there exists $v_0 \in V$ such that

$$D(u_0) < T(A(u_0) \rightarrow_1 B(v_0), \alpha) \rightarrow_2 B^*(v_0)$$

holds. Since \rightarrow_2 satisfies (C4), it follows that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq D(u_0) \rightarrow_2 B^*(v_0)$, and we have from the residual condition (10) that $\alpha \leq (A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (D(u_0) \rightarrow_2 B^*(v_0))$. Thus D is not a DFI-solution of FMT.

To sum up, A^* determined by (16) is the infimum of DFI-solutions of FMT. \square

Example 4.7. *It is easy to get that $I_L, I_G, I_{G_o}, I_{FD}, I_{Y_a}, I_{EP}$ are fuzzy implications satisfying (C4), (C11) and (C12). From Theorem 4.6, we obtain that if $\rightarrow_2 \in \{I_L, I_G, I_{G_o}, I_{FD}, I_{Y_a}, I_{EP}\}$ and (15) holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\}, u \in U$.*

Proposition 4.8. *Suppose that \rightarrow_2 is a fuzzy implication satisfying (C4), (C6), (C11) and (C12), and that $A \in F(U), B, B^* \in F(V), \alpha \in (0, 1]$, (15) holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\}, u \in U$.*

Proof. Suppose that T is the function residual to \rightarrow_2 . Since \rightarrow_2 satisfies (C6), we have from Theorem 4.6 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\} = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\}, u \in U$. \square

Proposition 4.9. *If \rightarrow_2 is an R-implication derived from left-continuous t-norm T , and $A \in F(U), B, B^* \in F(V), \alpha \in (0, 1]$, (15) holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\}, u \in U$.*

Proof. Since \rightarrow_2 is an R-implication derived from left-continuous t-norm T , it follows that \rightarrow_2 is a fuzzy implication satisfying (C4), (C5), (C6), (C11) and (C12) (see Remark 2.8). Thus we get by Proposition 4.8 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\} = \sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\}, u \in U$. \square

Example 4.10. *Note that $I_L, I_G, I_{G_o}, I_{FD}, I_{Y_a}, I_{EP}$ are R-implications derived from left-continuous t-norm. Thus it follows from Proposition 4.9 that if $\rightarrow_2 \in \{I_L, I_G, I_{G_o}, I_{FD}, I_{Y_a}, I_{EP}\}$ and (15) holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\}, u \in U$.*

Theorem 4.11. *If \rightarrow_2 is a fuzzy implication satisfying (C11) and (C12), and the following condition:*

$$(C13) \ I(a, b) = I(b', a') \quad (a, b \in [0, 1]),$$

and T is the function residual to \rightarrow_2 , and $A \in F(U)$, $B, B^ \in F(V)$, $\alpha \in (0, 1]$, (15) holds, then the infimum of DFI-solutions of FMT can be computed as follows:*

$$A^*(u) = \sup_{v \in V} \{T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha))\}, \quad u \in U, \quad (19)$$

where $T'(a, b) = 1 - T(a, b)$ ($a, b \in [0, 1]$).

Proof. Since the fuzzy implication \rightarrow_2 satisfies (C11) and (C12), we get that the residual condition (10) holds.

Let $H_1 = \{u \in U \mid A^*(u) = 1\}$ and $H_2 = \{u \in U \mid A^*(u) < 1\}$. Assume that $C \in F(U)$, and that $C(u) = 1$ for $u \in H_1$, and that $C(u) > A^*(u)$ for $u \in H_2$. We shall show that C is a DFI-solution of FMT, that is, the following inequality holds for any $u \in U, v \in V$:

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (C(u) \rightarrow_2 B^*(v)) < \alpha. \quad (20)$$

If $u \in H_1$, then it follows from (15) that $C(u) = 1$ satisfies (20) for any $v \in V$.

If $u \in H_2$, then it follows from (19) and $C(u) > A^*(u)$ that

$$C(u) > T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha)) \quad (21)$$

holds for any $v \in V$. Suppose, on the contrary, that (20) does not hold. Then there exist $u_0 \in U$ and $v_0 \in V$ such that $(A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (C(u_0) \rightarrow_2 B^*(v_0)) \geq \alpha$ holds (obviously $u_0 \in H_2$). Since \rightarrow_2 satisfies (C13), we get that $(A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 [(B^*(v_0))' \rightarrow_2 (C(u_0))'] \geq \alpha$, and it follows from residual condition (10) that $T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq (B^*(v_0))' \rightarrow_2 (C(u_0))'$, and $T((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha)) \leq (C(u_0))'$, thus

$$C(u_0) \leq T'((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha)),$$

which contradicts (21). Therefore (20) holds for any $u \in U, v \in V$. As a result, C is a DFI-solution of FMT.

Next, we shall check that A^* determined by (19) is the infimum of DFI-solutions of FMT. Assume that $D \in F(U)$, and that there exists $u_0 \in U$ such that $D(u_0) < A^*(u_0)$. We shall prove that D is not a DFI-solution of FMT. In fact, it follows from (19) that there exists $v_0 \in V$ such that

$$D(u_0) < T'((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha))$$

holds, thus we have $(D(u_0))' > T((B^*(v_0))', T(A(u_0) \rightarrow_1 B(v_0), \alpha))$. From the residual condition (10) and the fact that \rightarrow_2 satisfies (C13), we have that

$$T(A(u_0) \rightarrow_1 B(v_0), \alpha) \leq (B^*(v_0))' \rightarrow_2 (D(u_0))' = D(u_0) \rightarrow_2 B^*(v_0),$$

and $\alpha \leq (A(u_0) \rightarrow_1 B(v_0)) \rightarrow_2 (D(u_0) \rightarrow_2 B^*(v_0))$. Thus, D is not a DFI-solution of FMT.

To sum up, A^* determined by (19) is the infimum of DFI-solutions of FMT. \square

Example 4.12. It is easy to obtain that $I_L, I_{FD}, I_{GR}, I_R, I_{RR}$ are fuzzy implications satisfying (C11), (C12) and (C13). Thus it follows from Theorem 4.11 that if $\rightarrow_2 \in \{I_L, I_{FD}, I_{GR}, I_R, I_{RR}\}$ and (15) holds, then the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha))\}, u \in U$.

Lemma 4.13. [19] Suppose that I is an R -implication satisfying (C13) derived from left-continuous t -norm T , then the following formula holds:

$$(C14) \quad T'(b', a) = T'(a, b') = I(a, b) \quad (a, b \in [0, 1]).$$

Proposition 4.14. Suppose that \rightarrow_2 is an R -implication satisfying (C13) derived from left-continuous t -norm T , and $A \in F(U)$, $B, B^* \in F(V)$, $\alpha \in (0, 1]$, (15) holds, then ($u \in U$)

$$\sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\} = \sup_{v \in V} \{T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha))\}.$$

Proof. It follows from Lemma 4.13 that (C14) holds, and we get from Remark 2.8 that \rightarrow_2 satisfies (C6). T obviously is commutative, then it follows from Proposition 4.9 that the infimum of DFI-solutions of FMT is

$$\begin{aligned} A^*(u) &= \sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\} \\ &= \sup_{v \in V} \{T(\alpha, A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)\} \\ &= \sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\} \\ &= \sup_{v \in V} \{T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha))\}, \quad u \in U. \end{aligned}$$

□

Remark 4.15. If \rightarrow_2 is an R -implication satisfying (C13) derived from left-continuous t -norm, then Theorem 4.11 obviously holds. This moment, Proposition 4.14 demonstrates that the infimum of DFI-solutions of FMT achieved by Theorem 4.11 is equivalent to the one from Proposition 4.9.

Remark 4.16. For $\rightarrow_2 \in \{I_G, I_L, I_{FD}, I_{Go}, I_{Ya}, I_{EP}\}$, it follows from Theorem 4.6, Proposition 4.8 and Proposition 4.9 that the expressions of infimum of DFI-solutions of FMT achieved from Example 4.7 is the same as the one from Example 4.10. For $\rightarrow_2 \in \{I_L, I_{FD}\}$ (which is an R -implication satisfying (C13) derived from left-continuous t -norm), it follows from Proposition 4.14 that the expressions of infimum of DFI-solutions of FMT achieved by Example 4.10 and Example 4.12 (and also Example 4.7) are consistent.

Proposition 4.17. Suppose that $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_R, I_{Ya}, I_{EP}, I_{RR}\}$ in (8), and that (15) holds, then the infimum of DFI-solutions of FMT is as follows respectively ($u \in U$):

- (i) If \rightarrow_2 takes I_L , then $A^*(u) = \sup_{v \in V} \{B^*(v) - (A(u) \rightarrow_1 B(v)) + 2 - \alpha\}$;
- (ii) If \rightarrow_2 takes I_G , then $A^*(u) = \sup_{v \in V} \{B^*(v)\}$;
- (iii) If \rightarrow_2 takes I_{Go} , then $A^*(u) = \sup_{v \in V} \{B^*(v) / [\alpha \times (A(u) \rightarrow_1 B(v))]\}$;
- (iv) If \rightarrow_2 takes I_{FD} , then $A^*(u) = \sup_{v \in V} \{(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)\} \vee (1 - \alpha)$;

- (v) If \rightarrow_2 takes I_{GR} , then $A^*(u) = \sup_{v \in V} \{B^*(v)\}$;
- (vi) If \rightarrow_2 takes I_R , then $A^*(u) = \sup_{v \in V} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))']\}$;
- (vii) If \rightarrow_2 takes I_{Ya} , then $A^*(u) = \sup_{v \in V} \{1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - \alpha} - \sqrt{1 - (A(u) \rightarrow_1 B(v))})^2\}$;
- (viii) If \rightarrow_2 takes I_{EP} , then $A^*(u) = \sup_{v \in V} \{[2\phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)] / [\alpha + \phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)]\}$, where $\phi_{EP}(u, v) = B^*(v) \times [2 - (A(u) \rightarrow_1 B(v))] / [(A(u) \rightarrow_1 B(v)) + B^*(v) - B^*(v) \times (A(u) \rightarrow_1 B(v))]$;
- (ix) If \rightarrow_2 takes I_{RR} , then $A^*(u) = \sup_{v \in V} \{[\alpha' / ((A(u) \rightarrow_1 B(v)) \times (B^*(v))')] \vee [(A(u) \rightarrow_1 B(v))' / (B^*(v))'] \vee B^*(v)\}$.

Proof. If $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{Ya}, I_{EP}\}$, then it follows from Example 4.10 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\}$, $u \in U$. If $\rightarrow_2 \in \{I_{GR}, I_R, I_{RR}\}$, suppose that T is the function residual to \rightarrow_2 , then it follows from Example 4.12 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha))\}$, $u \in U$. Then we need to get the specific expression of $A^*(u)$. We only prove the cases of I_{FD}, I_{GR} as examples, the remainders can be proved similarly.

(iv) Suppose that \rightarrow_2 takes I_{FD} . It follows from (15) that we have $1 > B^*(v)$, $A(u) \rightarrow_1 B(v) > B^*(v)$, $[1 - (A(u) \rightarrow_1 B(v))] \vee B^*(v) < \alpha$ hold for any $u \in U, v \in V$. Thus, we obtain ($u \in U$):

$$\begin{aligned}
A^*(u) &= \sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\} \\
&= \sup_{v \in V} \{\alpha \rightarrow_2 [(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)]\} \\
&= \sup_{v \in V} \{(1 - \alpha) \vee (1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)\} \\
&= \sup_{v \in V} \{(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)\} \vee (1 - \alpha).
\end{aligned}$$

(v) Suppose that \rightarrow_2 takes I_{GR} . We get from Example 3.7 that T_{GR} is the function residual to I_{GR} . It follows from (15) that we get $1 > B^*(v)$, $A(u) \rightarrow_1 B(v) > 0$ hold for any $u \in U, v \in V$. So we have ($u \in U$):

$$\begin{aligned}
A^*(u) &= \sup_{v \in V} \{T'_{GR}((B^*(v))', T_{GR}(A(u) \rightarrow_1 B(v), \alpha))\} \\
&= \sup_{v \in V} \{1 - T_{GR}((B^*(v))', A(u) \rightarrow_1 B(v))\} \\
&= \sup_{v \in V} \{1 - (B^*(v))'\} = \sup_{v \in V} \{B^*(v)\}.
\end{aligned}$$

Theorem 4.18. *If $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, $A \in F(U)$, $B, B^* \in F(V)$, $\alpha \in (0, 1]$, (15) holds, then the condition, which the infimum A^* of DFI-solutions of FMT is the minimum, is as follows, respectively (for any $u \in U, v \in V$):*

- (i) Let \rightarrow_2 take I_L , $\sup_{v \in V} \{B^*(v) - (A(u) \rightarrow_1 B(v))\} > B^*(v) - (A(u) \rightarrow_1 B(v))$ holds;
- (ii) Let \rightarrow_2 take I_G , $\sup_{v \in V} \{B^*(v)\} > B^*(v)$ holds;

(iii) Let \rightarrow_2 take I_{Go} , $\sup_{v \in V} \{B^*(v)/(A(u) \rightarrow_1 B(v))\} > B^*(v)/(A(u) \rightarrow_1 B(v))$ holds;

(iv) Let \rightarrow_2 take I_{FD} , $\sup_{v \in V} \{(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)\} > [(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)] \vee (1 - \alpha)$ holds;

(v) Let \rightarrow_2 take I_{GR} , $\sup_{v \in V} \{B^*(v)\} > B^*(v)$ holds;

(vi) Let \rightarrow_2 take I_R , $\sup_{v \in V} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))']\} > \alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))']$ holds;

(vii) Let \rightarrow_2 take I_{Ya} , $\sup_{v \in V} \{1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - (A(u) \rightarrow_1 B(v))}) - \sqrt{1 - \alpha}\} > 1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - (A(u) \rightarrow_1 B(v))}) - \sqrt{1 - \alpha}$ holds;

(viii) Let \rightarrow_2 take I_{EP} , $\sup_{v \in V} \{[2\phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)] / [\alpha + \phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)]\} > [2\phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)] / [\alpha + \phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)]$ holds;

(ix) Let \rightarrow_2 take I_{RR} , $\sup_{v \in V} \{[\alpha' / ((A(u) \rightarrow_1 B(v)) \times (B^*(v))')] \vee [(A(u) \rightarrow_1 B(v))' / (B^*(v))'] \vee B^*(v)\} > [\alpha' / ((A(u) \rightarrow_1 B(v)) \times (B^*(v))')] \vee [(A(u) \rightarrow_1 B(v))' / (B^*(v))'] \vee B^*(v)$ holds.

Proof. Note that if the infimum A^* of DFI-solutions of FMT is a DFI-solution of FMT, then A^* is the minimum of DFI-solutions of FMT. Thus it is enough to prove that A^* is a DFI-solution of FMT (i.e., A^* should make (8) hold for any $u \in U, v \in V$). We still prove the cases of I_{FD}, I_{GR} as examples.

(iv) Let \rightarrow_2 take I_{FD} . It follows from Proposition 4.17 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)\} \vee (1 - \alpha)$. It follows from (15) that the following formula holds for any $u \in U, v \in V$:

$$1 > B^*(v), A(u) \rightarrow_1 B(v) > B^*(v), [1 - (A(u) \rightarrow_1 B(v))] \vee B^*(v) < \alpha. \quad (22)$$

By the condition given in (iv), we obtain that $A^*(u) > [(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)] \vee (1 - \alpha)$, so the following formula holds for any $u \in U, v \in V$:

$$A^*(u) > B^*(v), A(u) \rightarrow_1 B(v) > 1 - A^*(u), \alpha > 1 - A^*(u). \quad (23)$$

It follows from (22) and (23) that we have:

$$\begin{aligned} & (A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \\ &= (A(u) \rightarrow_1 B(v)) \rightarrow_2 [(1 - A^*(u)) \vee B^*(v)] \\ &= [1 - (A(u) \rightarrow_1 B(v))] \vee [(1 - A^*(u)) \vee B^*(v)] \\ &< \alpha. \end{aligned}$$

Thus A^* makes (8) hold for any $u \in U, v \in V$, and hence it is a DFI-solution of FMT.

(v) Let \rightarrow_2 take I_{GR} . We get from Proposition 4.17 that the infimum of DFI-solutions of FMT is $A^*(u) = \sup_{v \in V} \{B^*(v)\}$. It follows from (15) that $1 > B^*(v), A(u) \rightarrow_1 B(v) > 0$ hold for any $u \in U, v \in V$. By the condition given in (v), we obtain that $A^*(u) = \sup_{v \in V} \{B^*(v)\} > B^*(v)$ holds for any $u \in U, v \in V$. Thus

$$(A(u) \rightarrow_1 B(v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) = (A(u) \rightarrow_1 B(v)) \rightarrow_2 0 = 0 < \alpha.$$

So A^* is a DFI-solution of FMT. \square

The infimum A^*	\rightarrow_2
$\sup_{v \in V} \{T(A(u) \rightarrow_1 B(v), \alpha) \rightarrow_2 B^*(v)\}$	A fuzzy implication satisfying (C4), (C11) and (C12)
$\sup_{v \in V} \{(A(u) \rightarrow_1 B(v)) \rightarrow_2 (\alpha \rightarrow_2 B^*(v))\}$	A fuzzy implication satisfying (C4), (C6), (C11) and (C12)
$\sup_{v \in V} \{\alpha \rightarrow_2 [(A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v)]\}$	An R-implication from from left-continuous t-norm
$\sup_{v \in V} \{T'((B^*(v))', T(A(u) \rightarrow_1 B(v), \alpha))\}$	A fuzzy implication satisfying (C11), (C12) and (C13)

TABLE 3. Some Results (I) of the DFI-restriction Inference Algorithm for FMT

\rightarrow_2	The infimum A^*
I_L	$\sup_{v \in V} \{B^*(v) - (A(u) \rightarrow_1 B(v)) + 2 - \alpha\}$
I_G	$\sup_{v \in V} \{B^*(v)\}$
I_{Go}	$\sup_{v \in V} \{B^*(v) / [\alpha \times (A(u) \rightarrow_1 B(v))]\}$
I_{FD}	$\sup_{v \in V} \{(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)\} \vee (1 - \alpha)$
I_{GR}	$A^*(u) = \sup_{v \in V} \{B^*(v)\}$
I_R	$\sup_{v \in V} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))']\}$
I_{Ya}	$\sup_{v \in V} \{1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - \alpha} - \sqrt{1 - (A(u) \rightarrow_1 B(v))})^2\}$
I_{EP}	$\sup_{v \in V} \{[2\phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)] / [\alpha + \phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)]\}$
I_{RR}	$\sup_{v \in V} \{[\alpha' / ((A(u) \rightarrow_1 B(v)) \times (B^*(v))')] \vee [(A(u) \rightarrow_1 B(v))' / (B^*(v))'] \vee B^*(v)\}$

TABLE 4. Some Results (II) of the DFI-restriction Inference Algorithm for FMT

Table 3 concludes some results of the algorithm for FMT from Theorem 4.6, Proposition 4.8, Proposition 4.9 and Theorem 4.11. Moreover, Table 4 and Table 5 shows some results of the algorithm for FMT from Proposition 4.17 and Theorem 4.18.

Example 4.19. Let $U = V = [0, 1]$, $A(u) = (4 - u)/4$, $B(v) = (3 + v)/4$, $B^*(v) = (1 - v)/4$ and $\alpha = 1/2$, where $u \in U, v \in V$. Suppose that $\rightarrow_2 = I_R$, $\rightarrow_1 = I_L$ in the DFI-restriction inference algorithm for FMT. We now compute the infimum of DFI-solutions of FMT.

$$\begin{aligned}
A(u) \rightarrow_1 B(v) &= I_L(A(u), B(v)) = \begin{cases} 1 - \frac{4-u}{4} + \frac{3+v}{4}, & \text{if } \frac{4-u}{4} > \frac{3+v}{4} \\ 1, & \text{if } \frac{4-u}{4} \leq \frac{3+v}{4} \end{cases} \\
&= \begin{cases} \frac{3+u+v}{4}, & \text{if } u + v < 1 \\ 1, & \text{if } u + v \geq 1 \end{cases}.
\end{aligned}$$

\rightarrow_2	The condition which the infimum A^* is the minimum
I_L	$\sup_{v \in V} \{B^*(v) - (A(u) \rightarrow_1 B(v))\} > B^*(v) - (A(u) \rightarrow_1 B(v))$
I_G	$A^*(u) > B^*(v)$
I_{Go}	$\sup_{v \in V} \{B^*(v)/(A(u) \rightarrow_1 B(v))\} > B^*(v)/(A(u) \rightarrow_1 B(v))$
I_{FD}	$\sup_{v \in V} \{(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)\} > [(1 - (A(u) \rightarrow_1 B(v))) \vee B^*(v)] \vee (1 - \alpha)$
I_{GR}	$A^*(u) > B^*(v)$
I_R	$A^*(u) > \alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))']$
I_{Ya}	$A^*(u) > 1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - \alpha})^2$
I_{EP}	$A^*(u) > [2\phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)] / [\alpha + \phi_{EP}(u, v) - \alpha \times \phi_{EP}(u, v)]$
I_{RR}	$A^*(u) > [\alpha' / ((A(u) \rightarrow_1 B(v)) \times (B^*(v))')] \vee [(A(u) \rightarrow_1 B(v))' / (B^*(v))'] \vee B^*(v)$

TABLE 5. Some Results (III) of the DFI-restriction Inference Algorithm for FMT

For (15), we have

$$\begin{aligned}
& (A(u) \rightarrow_1 B(v)) \rightarrow_2 (1 \rightarrow_2 B^*(v)) \\
&= (A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v) \\
&= \begin{cases} 1 - \frac{3+u+v}{4} + \frac{3+u+v}{4} \times \frac{1-v}{4}, & \text{if } u+v < 1 \\ \frac{1-v}{4}, & \text{if } u+v \geq 1 \end{cases} \\
&= \begin{cases} \frac{7}{16} - \frac{3u+3v}{16} - \frac{v(3+u+v)}{16}, & \text{if } u+v < 1 \\ \frac{1-v}{4}, & \text{if } u+v \geq 1 \end{cases} \\
&< \alpha,
\end{aligned}$$

thus (15) holds. Then it follows from Proposition 4.17 that the infimum of DFI-solutions of FMT is as follows ($u \in U$):

$$\begin{aligned}
& A^*(u) = \sup_{v \in V} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))']\} \\
&= \sup_{v \in V} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))'] \mid u+v < 1\} \\
&\quad \vee \sup_{v \in V} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))'] \mid u+v \geq 1\} \\
&= \sup_{v \in [0,1]} \left\{ \frac{1}{2} \middle/ \left(\frac{3+u+v}{4} \times \frac{3+v}{4} \right) \middle| u+v < 1 \right\} \\
&\quad \vee \sup_{v \in [0,1]} \left\{ \frac{1}{2} \middle/ \frac{3+v}{4} \middle| u+v \geq 1 \right\} \\
&= \sup_{v \in [0,1]} \left\{ \frac{8}{(3+u+v) \times (3+v)} \middle| u+v < 1 \right\} \vee \sup_{v \in [0,1]} \left\{ \frac{2}{3+v} \middle| u+v \geq 1 \right\}.
\end{aligned}$$

(i) Suppose $u = 1$, then $\{v \in [0, 1] \mid u + v < 1\} = \emptyset$, and $0 \in \{v \in [0, 1] \mid u + v \geq 1\}$. Taking into account that $\frac{2}{3+v}$ is decreasing w.r.t. v , we get

$$A^*(u) = (\sup \emptyset) \vee \frac{2}{3} = 0 \vee \frac{2}{3} = \frac{2}{3} = \frac{8}{9+3u}.$$

(ii) Suppose $0 \leq u < 1$, then $0 \in \{v \in [0, 1] \mid u + v < 1\}$. Since $\frac{8}{(3+u+v) \times (3+v)}$, $\frac{2}{3+v}$ are decreasing w.r.t. v , we have

$$A^*(u) = \frac{8}{(3+u) \times 3} \vee \frac{2}{3+1-u} = \frac{8}{9+3u} \vee \frac{2}{4-u} = \frac{8}{9+3u},$$

where $\frac{8}{9+3u} > \frac{2}{4-u}$ since $0 \leq u < 1$.

Together we achieve

$$A^*(u) = \frac{8}{9+3u}, \quad u \in U.$$

Example 4.20. Let U, V, A, B, B^*, α be the same as in Example 4.19. Suppose that $\rightarrow_2 = I_R$, $\rightarrow_1 = I_R$ in the DFI-restriction inference algorithm for FMT, which degenerates into the full implication restriction inference algorithm for FMT (taking I_R). We now compute the infimum of DFI-solutions of FMT.

$$A(u) \rightarrow_1 B(v) = I_R(A(u), B(v)) = 1 - \frac{4-u}{4} + \frac{4-u}{4} \times \frac{3+v}{4} = \frac{12+u+v(4-u)}{16}.$$

For (15), we have

$$\begin{aligned} & (A(u) \rightarrow_1 B(v)) \rightarrow_2 (1 \rightarrow_2 B^*(v)) \\ &= (A(u) \rightarrow_1 B(v)) \rightarrow_2 B^*(v) \\ &= 1 - \frac{12+u+v(4-u)}{16} + \frac{12+u+v(4-u)}{16} \times \frac{1-v}{4} \\ &= \frac{7}{16} - \frac{3u+3v(4-u)}{64} - \frac{v(12+u+v(4-u))}{64} < \alpha, \end{aligned}$$

which means (15) holds. It follows from Proposition 4.17 that the infimum of DFI-solutions of FMT is as follows ($u \in U$):

$$\begin{aligned} A^*(u) &= \sup_{v \in V} \{\alpha' / [(A(u) \rightarrow_1 B(v)) \times (B^*(v))']\} \\ &= \sup_{v \in [0,1]} \left\{ \frac{1}{2} / \left(\frac{12+u+v(4-u)}{16} \times \frac{3+v}{4} \right) \right\} \\ &= \sup_{v \in [0,1]} \left\{ \frac{32}{[12+u+v(4-u)] \times (3+v)} \right\}. \end{aligned}$$

Because $\frac{32}{[12+u+v(4-u)] \times (3+v)}$ is decreasing w.r.t. v , we obtain

$$A^*(u) = \frac{32}{(12+u) \times 3} = \frac{32}{36+3u}.$$

	DFI-restriction inference algorithm for FMT	full implication restriction inference algorithm for FMT
Specific ones	81 kinds	9 kinds
Actual specific ones	64 kinds	8 kinds

TABLE 6. Specific Fuzzy Inference Algorithms for FMT

Remark 4.21. For the same U, V, A, B, B^*, α in Example 4.19 and Example 4.20, since

$$0 < u \leq 1 \Rightarrow \frac{8}{9+3u} < \frac{32}{36+3u},$$

$$u = 0 \Rightarrow \frac{8}{9+3u} = \frac{32}{36+3u},$$

the infimum of DFI-solutions of FMT (from the DFI-restriction inference algorithm) in Example 4.19 is smaller than the one (from the full implication restriction inference algorithm) in Example 4.20. From the viewpoint of DFI-restriction inference principle for FMT (which seeks the smallest A^* satisfying (8)), the DFI-restriction inference algorithm for FMT in Example 4.19 makes the inference closer, then it is better than the full implication restriction inference algorithm for FMT in Example 4.20.

Remark 4.22. Aiming at the set of fuzzy implications $\{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_R, I_{Ya}, I_{EP}, I_{RR}\}$, we get from the DFI-restriction inference algorithm that there are

$$9 * 9 = 81$$

kinds of specific fuzzy inference algorithms for FMT, in which

$$7 * 9 + 1 = 64$$

kinds of actual specific algorithms for FMT exist (noting that the expression is independent of \rightarrow_1 for the case $\rightarrow_2 \in \{I_G, I_{GR}\}$, by virtue of Table 4 and Table 5). But, from the full implication restriction inference algorithm, there are only 9 kinds of specific algorithms for FMT (including 8 kinds of actual specific algorithms for FMT), see Table 6. These imply that the DFI-restriction inference algorithm can obtain more and better specific algorithms for FMT (comparing with the full implication restriction inference algorithm), thus it is superior to the full implication restriction inference algorithm.

5. Conclusions

A new fuzzy inference strategy called the double fuzzy implications-based restriction inference algorithm (DFI-restriction inference algorithm for short) is put forward and investigated. First, new restriction inference principle is presented, which improves the full implication restriction inference principle; and then the definitions as well as existing conditions of solutions of the DFI-restriction inference algorithm are provided. Second, the supremum (or infimum) of related solutions is achieved, and the condition which the corresponding supremum is the maximum (or, the corresponding infimum is the minimum) is obtained for the FMP and FMT

problems, respectively. Third, the optimal solutions of the DFI-restriction inference algorithm are accomplished for several specific fuzzy implications. Lastly, it is found that the DFI-restriction inference algorithm is more reasonable than the full implication restriction inference algorithm, since the former can make the inference closer, and generate more, better specific inference algorithms. Such works would be helpful to applications of fuzzy inference as well as fuzzy system.

Acknowledgements. This work was supported by the National Natural Science Foundation of China (No. 61203077 and No. 61432004), the China Postdoctoral Science Foundation (No. 2012M521218 and No. 2014T70585), the Natural Science Foundation of Anhui Province (No. 1408085MKL15 and No. 1508085QF129).

REFERENCES

- [1] M. Baczynski and B. Jayaram, *Fuzzy implications* (Studies in Fuzziness and Soft Computing, Vol. 231), Springer, Berlin Heidelberg, 2008.
- [2] S. S. Dai, D. W. Pei and S. M. Wang, *Perturbation of fuzzy sets and fuzzy reasoning based on normalized Minkowski distances*, *Fuzzy Sets and Systems*, **189** (2012), 63-73.
- [3] D. Dubois and H. Prade, *Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions*, *Fuzzy Sets and Systems*, **40** (1991), 143-202.
- [4] J. Fodor, *On contrapositive symmetry of implications in fuzzy logic*, First European Congress on Fuzzy and Intelligent Technologies, Aachen, (1993), 1342-1348.
- [5] J. Fodor and M. Roubens, *Fuzzy Preference Modeling and Multicriteria Decision Support*, Kluwer Academic Publishers, Dordrecht, 1994.
- [6] S. Gottwald, *A treatise on many-valued logics*, Research Studies Press, Baldock, 2001.
- [7] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.
- [8] J. Hou, F. You and H. X. Li, *Fuzzy systems constructed by triple I algorithm and their response ability*, *Progress in Natural Science*, **15** (2005), 29-37.
- [9] B. Jayaram, *On the law of importation $(x \wedge y) \rightarrow z \equiv (x \rightarrow (y \rightarrow z))$ in fuzzy logic*, *IEEE Transactions on Fuzzy Systems*, **16** (2008), 130-144.
- [10] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.
- [11] H. X. Li, *Probability representations of fuzzy systems*, *Science in China. Series F*, **49** (2006), 339-363.
- [12] H. X. Li, F. You and J. Y. Peng, *Fuzzy controllers based on some fuzzy implication operators and their response functions*, *Progress in Natural Science*, **14** (2004), 15-20.
- [13] H. W. Liu, *Fully implicational methods for approximate reasoning based on interval-valued fuzzy sets*, *Journal of Systems Engineering and Electronics*, **21** (2010), 224-232.
- [14] H. W. Liu and G. J. Wang, *Continuity of triple I methods based on several implications*, *Computers and Mathematics with Applications*, **56** (2008), 2079-2087.
- [15] H. W. Liu and C. Li, *Fully implicational methods for interval-valued fuzzy reasoning with multi-antecedent rules*, *International Journal of Computational Intelligence Systems*, **4** (2011), 929-945.
- [16] M. Mas, M. Monserrat, J. Torrens and E. Trillas, *A survey on fuzzy implication functions*, *IEEE Transactions on Fuzzy Systems*, **15** (2007), 1107-1121.
- [17] D. W. Pei, *R_0 implication: characteristics and applications*, *Fuzzy Sets and Systems*, **131** (2002), 297-302.
- [18] D. W. Pei, *On the strict logic foundation of fuzzy reasoning*, *Soft Computing*, **8** (2004), 539-545.
- [19] D. W. Pei, *Unified full implication algorithms of fuzzy reasoning*, *Information Sciences*, **178** (2008), 520-530.
- [20] D. W. Pei, *Formalization of implication based fuzzy reasoning method*, *International Journal of Approximate Reasoning*, **53** (2012), 837-846.

- [21] J. Y. Peng, *Fully implicational triple I restriction algorithm for fuzzy reasoning based on some familiar implication operators*, Progress in Natural Science, **15** (2005), 539-546.
- [22] H. Seki and M. Mizumoto, *SIRMs connected fuzzy inference method adopting emphasis and suppression*, Fuzzy Sets and Systems, **215** (2013), 112-126.
- [23] S. J. Song, C. B. Feng and C. X. Wu, *Theory of restriction degree of triple I method with total inference rules of fuzzy reasoning*, Progress in Natural Science, **11** (2001), 58-66.
- [24] S. J. Song, C. B. Feng and E. S. Lee, *Triple I method of fuzzy reasoning*, Computers and Mathematics with Applications, **44** (2002), 1567-1579.
- [25] Y. M. Tang and X. P. Liu, *Differently implicational universal triple I method of (1, 2, 2) type*, Computers and Mathematics with Applications, **59** (2010), 1965-1984.
- [26] Y. M. Tang, F. J. Ren and Y. X. Chen, *Reversibility of FMT-universal triple I method based on I_L operator*, American Journal of Engineering and Technology Research, **11** (2011), 2763-2766.
- [27] Y. M. Tang, F. J. Ren and Y. X. Chen, *Universal triple I method and its application to textual emotion polarity recognition*, In: Third International Conference on Quantitative Logic and Soft Computing, Xi'an, China, (2012), 189-196.
- [28] L. X. Wang, *A course in fuzzy systems and control*, Prentice-Hall, Englewood Cliffs, NJ, 1997.
- [29] G. J. Wang, *Fully implicational triple I method for fuzzy reasoning*, Science in China. Series E, **29** (1999), 43-53.
- [30] G. J. Wang and L. Fu, *Unified forms of triple I method*, Computers and Mathematics with Applications, **49** (2005), 923-932.
- [31] G. J. Wang and H. J. Zhou, *Introduction to Mathematical Logic and Resolution Principle*, Science Press, Beijing and Alpha Science International Limited, Oxford, U.K., 2009.
- [32] L. A. Zadeh, *Outline of a new approach to the analysis of complex systems and decision processes*, IEEE Transactions on Systems Man and Cybernetics, **3** (1973), 28-44.
- [33] J. C. Zhang and X. Y. Yang, *Some properties of fuzzy reasoning in propositional fuzzy logic systems*, Information Sciences, **180** (2010), 4661-4671.

YIMING TANG*, SCHOOL OF COMPUTER AND INFORMATION, HEFEI UNIVERSITY OF TECHNOLOGY, HEFEI 230009, CHINA
E-mail address: tym608@163.com

XUEZHI YANG, SCHOOL OF COMPUTER AND INFORMATION, HEFEI UNIVERSITY OF TECHNOLOGY, HEFEI 230009, CHINA
E-mail address: xzyang@hfut.edu.cn

XIAOPING LIU, SCHOOL OF COMPUTER AND INFORMATION, HEFEI UNIVERSITY OF TECHNOLOGY, HEFEI 230009, CHINA
E-mail address: lxp@hfut.edu.cn

JUAN YANG, SCHOOL OF COMPUTER AND INFORMATION, HEFEI UNIVERSITY OF TECHNOLOGY, HEFEI 230009, CHINA
E-mail address: yangjuan6985@163.com

*CORRESPONDING AUTHOR