International Journal of Information Technology & Decision Making
Vol. 15, No. 5 (2016) 1235–1264

DOI: 10.1142/S0219622016500334

Variable Differently Implicational Inference for R- and S-Implications

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Published 19 July 2016

As a generalization of the compositional rule of inference (CRI) algorithm and the fully implicational algorithm, the differently implicational algorithm of fuzzy inference not only inherit the advantages of the fully implicational algorithm, but also has stronger practicability. Then, the variable differently implicational algorithm was proposed to make the current differently implicational algorithms compose a united whole. In this paper, the variable differently implicational algorithm is further researched focusing on the fuzzy modus tollens (FMT) problem. The differently implicational principle for FMT is improved. Moreover, the unified solutions of the variable differently implicational algorithm for FMT are accomplished for R- and S-implications. Following that, as an important index of fuzzy inference, the continuity of this algorithm is analyzed for main R- and S-implications, in which excellent performance is obtained. Finally, its optimal solutions as well as inference examples are provided for several specific R- and S-implications.

Keywords: Fuzzy inference; fuzzy modus tollens; fuzzy implication; compositional rule of inference; fully implicational algorithm.

1. Introduction

Fuzzy inference is an advanced computing framework based on the concepts of fuzzy set, fuzzy if-then rule, approximate inference, which has significant application value

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in fuzzy control, pattern recognition, machine learning, affective computing and so on.[1–3] It has two fundamental problems, which are called fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) which are expressed as follows:

$$\text{FMP : from } A \to B \quad \text{and} \quad A^*, \text{ compute } B^*, \tag{1}$$

$$\text{FMT : from } A \to B \quad \text{and} \quad B^*, \text{ compute } A^*. \tag{2}$$

Here, $A, A^* \in F(U)$, $B, B^* \in F(V)$, in which $F(U), F(V)$ respectively denotes the set of fuzzy subsets of U, V. The compositional rule of inference (CRI) algorithm proposed by Zadeh is the classical and widely used algorithm.[4,5] As its modification, the fully implicational algorithm was proposed by Wang in 1999.[6,7] Its optimal solution is the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) making

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \geq \alpha \tag{3}$$

hold for any $u \in U$, $v \in V$ (in which $\alpha \in (0, 1]$, and $\to$ is a fuzzy implication). It is verified that the fully implicational algorithm has many wonderful merits, which lie in its strict logic basis, reversibility properties, the ability of pointwise optimization and so on.[8–11] However, it is not perfect from the view of some kind of fuzzy system owing to its weak response ability and practicability.[12–14]

Aiming at such problem, in Ref. 15, the fully implicational algorithm was generalized to the differently implicational algorithm. Its solution is the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) letting

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \geq \alpha, \tag{4}$$

hold for any $u \in U, v \in V$ ($\alpha \in (0, 1]$). The differently implicational algorithm also takes the CRI algorithm as its special case.[15] In Ref. 16, the differently implicational algorithm was discussed for FMP, in which reversibility properties and more general fuzzy systems were researched, and it was applied to emotion polarity recognition. In Ref. 17, the differently implicational algorithm was investigated for FMP and FMT from the meaning of fuzzy reasoning as well as fuzzy controller, in which the R-implications, (0,1)-implications, as well as the expansion, reduction and other type operators were respectively employed. Its optimal solutions were obtained with verifying its reversibility, and the response abilities of corresponding fuzzy controllers were researched. In Ref. 18, it was found that 190 fuzzy systems via the differently implicational algorithm could be used in practical systems, while 19 fuzzy systems via the CRI method and two ones via the fully implicational algorithm were practicable. So the differently implicational algorithm has larger effective choosing space, which can achieve more usable fuzzy systems comparing with the fully implicational algorithm and the CRI algorithm. To sum up, the differently implicational algorithm not only inherit the advantages of the fully implicational algorithm, but also has stronger practicability.

Furthermore, to reveal the inherent essence of current differently implicational algorithms, the variable differently implicational algorithm was put forward in

Ref. 19, which aimed at

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \geq \alpha(u, v). \tag{5}$$

Here, $\alpha(u, v)$ is a variable parameter ($\alpha(u, v) \in [0, 1]$). Focusing on the FMP problem, the optimal solutions of the variable differently implicational algorithm were analyzed and obtained in Ref. 19. This algorithm makes the current differently implicational algorithms compose a united whole.

As mentioned above, the FMP and FMT problems are two fundamental problems of fuzzy inference. As a result, it is valuable to research the variable differently implicational algorithm for the FMT problem, which constructs the main aim of this paper.

For a fuzzy inference method, it is hopeful that small input deviation does not lead to the huge deviation of the reasoning result. This is called the index of continuity, which is recognized as an important index for fuzzy reasoning. Here, we will discuss the continuity of the variable differently implicational algorithm for FMT.

This paper is organized as follows. Section 2 is the preliminaries. Section 3 gives the basic principle, definitions and general results of the variable differently implicational algorithm for FMT. Section 4 establishes the unified forms of the variable differently implicational algorithm for FMT, where $\to_2$ takes an R-implication. In Sec. 5, we research the variable differently implicational algorithm for FMT aiming at the case of S-implications. In Sec. 6, we analyze the continuity of the variable differently implicational algorithm for FMT in which $\to_2$ takes an R-implication or S-implication. Section 7 provides two specific computing examples. Lastly, Sec. 8 draws the conclusions for the whole paper.

2. Preliminaries

There are several definitions of fuzzy implications. But all of these definitions need to maintain consistency with classical logic. So the basic one as Definition 2.1 is employed, which is also chosen by many papers.[2,9,17,19]

Definition 2.1. A fuzzy implication on $[0, 1]$ is a function $I : [0, 1]^2 \to [0, 1]$ satisfying

$(P1)$ $I(0, 0) = I(0, 1) = I(1, 1) = 1,\ I(1, 0) = 0.$

$I(a, b)$ can also be denoted as $a \to b$ ($a, b \in [0, 1]$).

In Ref. 20, the following definition is provided.

Definition 2.2. Suppose that T, I are two $[0, 1]^2 \to [0, 1]$ functions. (T, I) is called a residual pair or, T and I are residual to each other, if the following residuation condition holds:

$$T(a, b) \leq c \Longleftrightarrow b \leq I(a, c)\,(a, b, c \in [0, 1]). \tag{6}$$

In Refs. 19 and 21, the following proposition is shown.

Proposition 2.1. *If I is a fuzzy implication which satisfies*

($P2$) $I(a,1) = 1,\ a \in [0,1]$,
($P3$) $I(a,b) \geq I(a,c)$ *if* $b \geq c,\ a,b,c \in [0,1]$,
($P4$) $I(a,b)$ *is right-continuous with respect to* (*w.r.t. for short*) $b,\ a,b \in [0,1]$,

then the function $T : [0,1]^2 \to [0,1]$ *expressed as*

$$T(a,b) = \inf\{x \in [0,1] \mid b \leq I(a,x)\}, a, b \in [0,1]$$

is residual to I.

In Ref. 5, the following four definitions are given.

Definition 2.3. A function $T : [0,1]^2 \to [0,1]$ is called a t-norm if T is associative, commutative, increasing and satisfies $T(1,a) = a$ ($a \in [0,1]$).

Definition 2.4. A function $T : [0,1]^2 \to [0,1]$ is called a t-conorm if T is associative, commutative, increasing and satisfies $S(0,a) = a$ ($a \in [0,1]$).

Definition 2.5. A fuzzy negation is a decreasing function $N : [0,1] \to [0,1]$ satisfying

$$N(0) = 1, \quad N(1) = 0.$$

A fuzzy negation N is said to be strong if $N(N(a)) = a$ holds ($a \in [0,1]$).

$N_s(a) = 1 - a$ ($a \in [0,1]$) is said to be the standard negation on $[0,1]$, which is a strong negation.

Definition 2.6. The dual of a t-norm T on $[0,1]$ w.r.t. a strong negation N is the function T_N which is expressed as ($a, b \in [0,1]$)

$$T_N(a,b) = N(T(N(a), N(b))).$$

The dual of a t-conorm S on $[0,1]$ w.r.t. a strong negation N is the function S_N which is computed as ($a, b \in [0,1]$)

$$S_N(a,b) = N(S(N(a), N(b))).$$

It is noted that T_N is a t-conorm and S_N is a t-norm.

Nowadays, R-implications and S-implications are two kinds of important fuzzy implications,[22–24] see the following definitions.[2]

Definition 2.7. A function $I : [0,1]^2 \to [0,1]$ is said to be an R-implication, if there exists a left-continuous t-norm T such that ($a, b \in [0,1]$)

$$I(a,b) = \sup\{x \in [0,1] \mid T(a,x) \leq b\}. \tag{7}$$

Moreover, if an R-implication is generated from T, then it is represented by I_T.

In Refs. 24 and 25, the following lemma is shown.

Lemma 2.1. *Suppose that T is a left-continuous t-norm on $[0,1]$, and that I an R-implication obtained from (7). Then, (T, I) is a residual pair, and I satisfies $(P3)$, $(P4)$ as well as*

$(P5)$ $I(a, c) \geq I(b, c)$ *if* $a \leq b$,
$(P6)$ $I(a, b)$ *is left-continuous w.r.t.* a,
$(P7)$ $I(1, a) = a$,
$(P8)$ $I(a, I(b, c)) = I(b, I(a, c))$,
$(P9)$ $I(T(a, b), c) = I(a, I(b, c))$,
$(P10)$ $a \leq b \Longleftrightarrow I(a, b) = 1$,
$(P11)$ $a \leq I(b, c) \Longleftrightarrow b \leq I(a, c)$,
$(P12)$ $I(\sup_{x \in X} x, a) = \inf_{x \in X} I(x, a)$,
$(P13)$ $I(a, \inf_{x \in X} x) = \inf_{x \in X} I(a, x)$,

where $a, b, c \in [0,1]$ *and* $X \subset [0,1]$, $X \neq \emptyset$.

In Ref. 24, the following two definitions are provided.

Definition 2.8. A function $I : [0,1]^2 \rightarrow [0,1]$ is called an S-implication if there exist a t-conorm S and a strong negation N such that

$$I(a, b) = S(N(a), b), \quad a, b \in [0,1]. \tag{8}$$

Furthermore, if an S-implication is obtained from S and N, then it is denoted by $I_{S,N}$.

Proposition 2.2. *Suppose that I is an S-implication denoted from a t-conorm S and a strong negation N, then I is a fuzzy implication satisfying* $(P1)$, $(P2)$, $(P3)$, $(P5)$, $(P7)$, $(P8)$ *as well as*

$(P14)$ $I(a, b) = I(N(b), N(a)), \quad a, b \in [0,1]$.

In Ref. 26, the following definition is shown.

Definition 2.9. Let Z be any nonempty set, then partial order relation $\leq_F$ on $F(Z)$ is defined as:

$$A \leq_F B \Longleftrightarrow A(z_0) \leq B(z_0) \, (z_0 \in Z; A, B \in F(Z)).$$

Lemma 2.2. $\langle F(Z), \leq_F \rangle$ *is a complete lattice.*

In what follows, we denote $R_1(u, v) = A(u) \rightarrow_1 B(v)$, and $a' = 1 - a$ $(a \in [0,1])$ and $A'(x) = 1 - A(x)$ for any fuzzy set A, and finally $T'(a, b) = 1 - T(a, b)$ for any mapping $T : [0,1]^2 \rightarrow [0,1]$.

3. Fundamental Properties of the Variable Differently Implicational Algorithm for FMT

Aiming at the FMT problem expressed as (2), we can achieve the following principle for the variable differently implicational algorithm:

Variable differently implicational principle for FMT: The conclusion A^* of FMT problem (2) is the largest fuzzy set satisfying (5) in $\langle F(U), \leq_F \rangle$.

It is evident that such differently implicational principle for FMT improves the previous one in Ref. 15. The variable differently implicational algorithm for FMT is also called the $\alpha(u,v)$-FMT-differently implicational algorithm for short.

Definition 3.1. Let $A \in F(U)$, $B, B^* \in F(V)$, if A^* (in $\langle F(U), \leq_F\rangle$) makes (5) hold for any $u \in U, v \in V$, then A^* is called an $\alpha(u,v)$-FMT-differently implicational solution ($\alpha(u,v)$-FMT-solution for short).

Definition 3.2. Suppose that $A \in F(U)$, $B, B^* \in F(V)$, and that nonempty set $\mathbb{F}_{\alpha(u,v)}$ is the set of all $\alpha(u,v)$-FMT-solutions, and finally that C^* (in $\langle F(U), \leq_F\rangle$) is the supremum of $\mathbb{F}_{\alpha(u,v)}$. Then, C^* is called an $\alpha(u,v)$-SupT-quasi solution. And, if C^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$, then C^* is also called an $\alpha(u,v)$-MaxT-solution.

The following Proposition 3.1 shows a basic property of $\alpha(u,v)$-FMT-solution.

Proposition 3.1. *Suppose that $\rightarrow_2$ is a fuzzy implication satisfying $(P3)$ and $(P5)$, and that C_1 is an $\alpha(u,v)$-FMT-solution, and finally that $C_2 \leq_F C_1$ (in which $C_1, C_2 \in \langle F(U), \leq_F\rangle$). Then, C_2 is an $\alpha(u,v)$-FMT-solution.*

Proof. Because C_1 is an $\alpha(u,v)$-FMT-solution, it follows that

$$R_1(u,v) \rightarrow_2 (C_1(u) \rightarrow_2 B^*(v)) \geq \alpha(u,v)$$

holds for any $u \in U, v \in V$. Since $C_2 \leq_F C_1$ and $\rightarrow_2$ satisfies $(P3)$ and $(P5)$, we have

$$C_2(u) \rightarrow_2 B^*(v) \geq C_1(u) \rightarrow_2 B^*(v)$$

and

$$\begin{aligned} &R_1(u,v) \rightarrow_2 (C_2(u) \rightarrow_2 B^*(v)) \\ &\geq R_1(u,v) \rightarrow_2 (C_1(u) \rightarrow_2 B^*(v)) \geq \alpha(u,v) \end{aligned}$$

holds for any $u \in U, v \in V$. Consequently, C_2 is also an $\alpha(u,v)$-FMT-solution. □

Remark 3.1. Suppose that $\rightarrow_2$ satisfies $(P3)$ and $(P5)$. For (5), once there exists an $\alpha(u,v)$-FMT-solution A^*, then every fuzzy set C which is smaller than A^* ($C \in F(U)$), will be an $\alpha(u,v)$-FMT-solution. Thus, there are many $\alpha(u,v)$-FMT-solutions, including

$$A^*(u) \equiv 0 (u \in U).$$

This last is a special solution, for which (5) always holds no matter what $A \rightarrow_1 B$ and B^* are adopted. Thus, when the optimal $\alpha(u,v)$-FMT-solution exists, it should be the largest one; in other words, it should be the supremum.

Assume that the maximum of

$$R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v))$$

for FMT at every point (u,v) is $M_T(u,v)$. It is easy to prove Lemma 3.1.

Lemma 3.1. *Let* $\rightarrow$ *be a fuzzy implication satisfying* $(P3)$ *and* $(P5)$, *then* $a \rightarrow 1 = 1$, $0 \rightarrow b = 1 (a, b \in [0, 1])$.

The following Proposition 3.2 provides the maximum value of $R_1(u, v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v))$.

Proposition 3.2. *If* $\rightarrow_2$ *is a fuzzy implication satisfying* $(P3)$, $(P5)$, *then* $M_T(u, v) = 1$ $(u \in U, v \in V)$.

Proof. It follows from Lemma 3.1 that

$$R_1(u, v) \rightarrow_2 (0 \rightarrow_2 B^*(v)) = 1$$

obviously holds for any $u \in U, v \in V$. Take $A^*(u) \equiv 0 (u \in U)$, then

$$R_1(u, v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) = R_1(u, v) \rightarrow_2 (0 \rightarrow_2 B^*(v)) = 1.$$

Thus, $M_T(u, v) \geq 1$, which implies $L(u, v) = 1$ holds for any $u \in U, v \in V$. □

To guarantee (5) holds, we always assume in this subsection that $\alpha(u, v) \leq M_T(u, v)$ holds for any $u \in U, v \in V$. Especially, if $\rightarrow_2$ satisfies $(P3)$ and $(P5)$, then $M_T(u, v) = 1$, which means $\alpha(u, v) \leq M_T(u, v) = 1$ naturally holds for any $u \in U, v \in V$.

We know from Lemma 2.2 that $\langle F(U), \leq_F \rangle$ is a complete lattice. So the $\alpha(u, v)$-SupT-quasi solution (i.e., the supremum of $\mathbb{F}_{\alpha(u,v)}$) uniquely exists because the nonempty set $\mathbb{F}_{\alpha(u,v)} \subset F(U)$.

4. $\boldsymbol{\alpha(u,v)}$-FMT-Differently Implicational Algorithm for R-Implications

The following Proposition 4.1 analyzes the relationship between the $\alpha(u, v)$-SupT-quasi solution and the $\alpha(u, v)$-MaxT-solution.

Proposition 4.1. *If the fuzzy implication* $\rightarrow_2$ *satisfies* $(P4)$, $(P5)$ *and* $(P6)$, *then the* $\alpha(u, v)$*-SupT-quasi solution* A^* *is the* $\alpha(u, v)$*-MaxT-solution.*

Proof. Taking into account that the $\alpha(u, v)$-SupT-quasi solution $A^* = \sup \mathbb{F}_{\alpha(u,v)}$, it is enough to verify that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. It is obvious that

$$\mathbb{F}_{\alpha(u,v)} = \{C^* \in F(U) \mid R_1(u, v) \rightarrow_2 (C^*(u) \rightarrow_2 B^*(v)) \geq \alpha(u, v), u \in U, v \in V\}.$$

Suppose, on the contrary, that $A^* \notin \mathbb{F}_{\alpha(u,v)}$, then there exist fuzzy sets $A_1, A_2, \ldots$ in $\mathbb{F}_{\alpha(u,v)}$ such that

$$\lim_{n \to \infty} A_n(u) = A^*(u), \quad u \in U. \tag{9}$$

Noting that $A_1, A_2, \ldots \in \mathbb{F}_{\alpha(u,v)}$, we have $(n = 1, 2, \ldots, u \in U, v \in V)$:

$$R_1(u, v) \rightarrow_2 (A_n(u) \rightarrow_2 B^*(v)) \geq \alpha(u, v). \tag{10}$$

Since $A^* = \sup \mathbb{F}_{\alpha(u,v)}$, we obtain $A^*(u) \geq A_n(u)$ $(u \in U, n = 1, 2, \ldots)$, and it follows from (9) that $A^*(u)$ is the left limit of

$$\{A_n(u) |\, n = 1, 2, \ldots\}(u \in U).$$

This implies (noting that $\rightarrow_2$ satisfies $(P6)$)

$$\lim_{n\to\infty} \{A_n(u) \rightarrow_2 B^*(v)\} = A^*(u) \rightarrow_2 B^*(v). \tag{11}$$

Because $A^*(u) \geq A_n(u)$ and $\rightarrow_2$ satisfies $(P5)$, we have

$$A^*(u) \rightarrow_2 B^*(v) \leq A_n(u) \rightarrow_2 B^*(v)$$

$(u \in U,\ v \in V,\ n = 1, 2, \ldots)$. So we know that $A^*(u) \rightarrow_2 B^*(v)$ is the right limit of

$$\{A_n(u) \rightarrow_2 B^*(v) |\, n = 1, 2, \ldots\}.$$

Noting that $\rightarrow_2$ satisfies $(P4)$, it follows from (10) and (11) that we obtain $(u \in U, v \in V)$

$$\begin{aligned} \alpha(u,v) &\leq \lim_{n\to\infty} \{R_1(u,v) \rightarrow_2 (A_n(u) \rightarrow_2 B^*(v))\} \\ &= R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)). \end{aligned}$$

So $A^* \in \mathbb{F}_{\alpha(u,v)}$, a contradiction. Thus, $A^* \in \mathbb{F}_{\alpha(u,v)}$, and thus A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. □

It follows from Proposition 4.1 and Lemma 2.1 that we can get Theorem 4.1. It shows the relationship between the $\alpha(u,v)$-SupT-quasi solution and the $\alpha(u,v)$-MaxT-solution for the R-implication.

Theorem 4.1. *If $\rightarrow_2$ is an R-implication, then the $\alpha(u,v)$-SupT-quasi solution A^* is the $\alpha(u,v)$-MaxT-solution.*

The following Theorem 4.2 provides the $\alpha(u,v)$-MaxT-solution for the R-implication.

Theorem 4.2. *Suppose that $\rightarrow_2$ is an R-implication, then the $\alpha(u,v)$-MaxT-solution is as follows*:

$$A^*(u) = \inf_{v\in V} \{T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B^*(v)\}, \quad u \in U. \tag{12}$$

Proof. Since the R-implication $\rightarrow_2$ satisfies $(P8)$, $(P9)$ and $(P10)$, it follows that (5) is equivalent to the following formulas $(u \in U, v \in V)$:

$$\begin{aligned} &\alpha(u,v) \leq R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)), \\ &\alpha(u,v) \rightarrow_2 (R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v))) = 1, \\ &T(\alpha(u,v), R_1(u,v)) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) = 1, \\ &A^*(u) \rightarrow_2 (T(\alpha(u,v), R_1(u,v)) \rightarrow_2 B^*(v)) = 1, \\ &A^*(u) \leq T(\alpha(u,v), R_1(u,v)) \rightarrow_2 B^*(v). \end{aligned}$$

Hence, we get from Definition 3.2 that the $\alpha(u, v)$-MaxT-solution is expressed as follows (noting that T is commutative):

$$\begin{aligned} A^*(u) &= \inf_{v \in V} \{ T(\alpha(u, v), R_1(u, v)) \rightarrow_2 B^*(v)\} \\ &= \inf_{v \in V} \{ T(R_1(u, v), \alpha(u, v)) \rightarrow_2 B^*(v)\}, \quad u \in U. \end{aligned}$$

□

The following Theorem 4.3 provides another computing formula for the R-implication.

Theorem 4.3. *Suppose that* $\rightarrow_2$ *is an R-implication, then the* $\alpha(u, v)$*-MaxT-solution is as follows* ($u \in U$):

$$A^*(u) = \inf_{v \in V} \{R_1(u, v) \rightarrow_2 (\alpha(u, v) \rightarrow_2 B^*(v))\}. \tag{13}$$

Proof. It follows from Theorem 4.2 that the $\alpha(u, v)$-MaxT-solution is

$$A^*(u) = \inf_{v \in V} \{ T(R_1(u, v), \alpha(u, v)) \rightarrow_2 B^*(v)\}, \ u \in U,$$

where T is the mapping residual to $\rightarrow_2$. Note that the R-implication $\rightarrow_2$ satisfies $(P9)$, thus we have

$$\begin{aligned} A^*(u) &= \inf_{v \in V} \{ T(R_1(u, v), \alpha(u, v)) \rightarrow_2 B^*(v)\} \\ &= \inf_{v \in V} \{R_1(u, v) \rightarrow_2 (\alpha(u, v) \rightarrow_2 B^*(v))\}, \quad u \in U. \end{aligned}$$

□

The following fuzzy implications are R-implications, which include Lukasiewicz implication I_L, Gödel implication I_G, Goguen implication I_{Go}, I_0 implication[6,27] (which is also called I_{FD}, see Ref. 23), and $I_{ep}, I_{y-0.5}$.[15,28]

$$I_L(a, b) = \begin{cases} 1, & a \leq b, \\ a' + b, & a > b, \end{cases}$$

$$I_G(a, b) = \begin{cases} 1, & a \leq b, \\ b, & a > b, \end{cases}$$

$$I_{Go}(a, b) = \begin{cases} 1, & a = 0, \\ (b/a) \wedge 1, & a \neq 0, \end{cases}$$

$$I_0(a, b) = \begin{cases} 1, & a \leq b, \\ a' \vee b, & a > b, \end{cases}$$

$$I_{ep}(a, b) = \begin{cases} 1, & a \leq b, \\ (2b - ab)/(a + b - ab), & a > b, \end{cases}$$

$$I_{y-0.5}(a, b) = \begin{cases} 1, & a \leq b, \\ 1 - (\sqrt{1 - b} - \sqrt{1 - a})^2, & a > b. \end{cases}$$

It is easy to get Lemma 4.1.

Lemma 4.1. *The mapping corresponding to the R-implications $I_L, I_G, I_{Go}, I_0, I_{ep}$, $I_{y-0.5}$ in residual pairs are as follows, respectively.*

$$T_L(a,b) = \begin{cases} a+b-1, & a+b>1, \\ 0, & a+b \le 1, \end{cases}$$
$$T_G(a,b) = a \wedge b,$$
$$T_{Go}(a,b) = a \times b,$$
$$T_0(a,b) = \begin{cases} a \wedge b, & a+b>1, \\ 0, & a+b \le 1, \end{cases}$$
$$T_{ep}(a,b) = ab/(2-a-b+ab),$$
$$T_{y-0.5}(a,b) = \begin{cases} 1-(f(a,b))^2, & f(a,b) \le 1, \\ 0, & f(a,b) > 1, \end{cases} \quad \text{where } f(a,b) = \sqrt{1-a}+\sqrt{1-b}.$$

For the R-implications mentioned above, we can achieve Proposition 4.2. It gives the unified $\alpha(u,v)$-MaxT-solution for some specific R-implications.

Proposition 4.2. *If $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_0, I_{ep}, I_{y-0.5}\}$, and T is the mapping residual to $\rightarrow_2$, then the $\alpha(u,v)$-MaxT-solution is expressed as (12) or (13).*

The following Proposition 4.3 shows the specific $\alpha(u,v)$-MaxT-solution for these R-implications.

Proposition 4.3. *If the R-implication $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_0, I_{ep}, I_{y-0.5}\}$, then the specific form of $\alpha(u,v)$-MaxT-solution is as follows, respectively ($u \in U$):*

(i) *If $\rightarrow_2$ takes I_L, then*

$$A^*(u) = \inf_{v \in F_u} \{2 - R_1(u,v) - \alpha(u,v) + B^*(v)\},$$

where $F_u = \{v \in V \mid R_1(u,v) + \alpha(u,v) - 1 > B^*(v)\}$.

(ii) *If $\rightarrow_2$ takes I_G, then*

$$A^*(u) = \inf_{v \in F_u} \{B^*(v)\},$$

where $F_u = \{v \in V \mid R_1(u,v) \wedge \alpha(u,v) > B^*(v)\}$.

(iii) *If $\rightarrow_2$ takes I_{Go}, then*

$$A^*(u) = \inf_{v \in F_u} \{B^*(v)/(R_1(u,v) \times \alpha(u,v))\},$$

where $F_u = \{v \in V \mid R_1(u,v) \times \alpha(u,v) > B^*(v)\}$.

(iv) *If $\rightarrow_2$ takes I_0, then*

$$A^*(u) = \inf_{v \in F_u} \{(R_1(u,v))' \vee (\alpha(u,v))' \vee B^*(v)\},$$

where $F_u = \{v \in V \mid R_1(u,v) + \alpha(u,v) > 1, R_1(u,v) \wedge \alpha(u,v) > B^*(v)\}$.

(v) *If* $\to_2$ *takes* I_{ep}, *then*

$$A^*(u) = \inf_{v \in F_u} \left\{ \frac{2B^*(v) - \varphi_{ep}(u,v) \times B^*(v)}{\varphi_{ep}(u,v) + B^*(v) - \varphi_{ep}(u,v) \times B^*(v)} \right\},$$

where $F_u = \{v \in V |\ \varphi_{ep}(u,v) > B^*(v)\}$.

(vi) *If* $\to_2$ *takes* $I_{y-0.5}$, *then*

$$A^*(u) = \inf_{v \in F_u} \{1 - (\sqrt{1 - B^*(v)} - \sqrt{1 - R_1(u,v)} - \sqrt{1 - \alpha(u,v)})^2\},$$

where $F_u = \{v \in V |\ \sqrt{1 - B^*(v)} > \sqrt{1 - R_1(u,v)} + \sqrt{1 - \alpha(u,v)}\ \}$.

Proof. For the R-implication $\to_2 \in \{I_L, I_G, I_{Go}, I_0, I_{ep}, I_{y-0.5}\}$, it follows from Theorem 4.2 that the $\alpha(u,v)$-MaxT-solution is

$$A^*(u) = \inf_{v \in V} \{T(R_1(u,v), \alpha(u,v)) \to_2 B^*(v)\},\ \ u \in U,$$

where T is the t-norm residual to $\to_2$. We only prove the case of I_L as an example, the remainders can be proved similarly.

Let $\to_2$ be I_L. We know from Lemma 4.1 that $T_L(a,b) = \begin{cases} a+b-1, & a+b>1 \\ 0, & a+b \le 1 \end{cases}$ is the mapping residual to I_L. So, we get $(u \in U)$

$$A^*(u) = \inf_{v \in V} \{I_L(T_L(R_1(u,v), \alpha(u,v)), B^*(v))\}.$$

Suppose $R_1(u,v) + \alpha(u,v) \le 1$ holds, then $T_L(R_1(u,v), \alpha(u,v)) = 0$, and thus

$$I_L(T_L(R_1(u,v), \alpha(u,v)), B^*(v)) = 1.$$

Suppose $R_1(u,v) + \alpha(u,v) > 1$ holds, then

$$T_L(R_1(u,v), \alpha(u,v)) = R_1(u,v) + \alpha(u,v) - 1.$$

If $R_1(u,v) + \alpha(u,v) - 1 \le B^*(v)$, then

$$I_L(T_L(R_1(u,v), \alpha(u,v)), B^*(v)) = 1;$$

otherwise,

$$\begin{aligned} I_L(T_L(R_1(u,v), \alpha(u,v)), B^*(v)) \\ &= 1 - (R_1(u,v) + \alpha(u,v) - 1) + B^*(v) \\ &= 2 - R_1(u,v) - \alpha(u,v) + B^*(v). \end{aligned}$$

Denote $F_u = \{v \in V |\ R_1(u,v) + \alpha(u,v) > 1,\ R_1(u,v) + \alpha(u,v) - 1 > B^*(v)\}$. It is easy to know that

$$R_1(u,v) + \alpha(u,v) - 1 > B^*(v)$$

implies

$$R_1(u,v) + \alpha(u,v) > 1.$$

Thus,

$$F_u = \{v \in V|\ R_1(u,v) + \alpha(u,v) - 1 > B^*(v)\}.$$

So, we obtain

$$\begin{aligned} A^*(u) = &\inf_{v\in V}\{I_L(T_L(R_1(u,v),\alpha(u,v)),B^*(v))\} \\ &= [\inf_{v\in F_u}\{I_L(T_L(R_1(u,v),\alpha(u,v)),B^*(v))\}] \\ &\quad \wedge [\inf_{v\in V-F_u}\{I_L(T_L(R_1(u,v),\alpha(u,v)),B^*(v))\}] \\ &= [\inf_{v\in F_u}\{2 - R_1(u,v) - \alpha(u,v) + B^*(v)\}] \\ &\quad \wedge [\inf_{v\in V-F_u}\{1\}], \quad u \in U. \end{aligned}$$

If $V - F_u = \emptyset$, then $\inf_{v\in V-F_u}\{1\} = \inf\emptyset = 1$; otherwise we also have $\inf_{v\in V-F_u}\{1\} = 1$. Therefore,

$$A^*(u) = \inf_{v\in F_u}\{2 - R_1(u,v) - \alpha(u,v) + B^*(v)\}(u \in U).$$

□

5. α(u,v)-FMT-Differently Implicational Algorithm for S-Implications

For the case of S-implication, the following Proposition 5.1 gives an equivalent form for (5).

Proposition 5.1. *Let* $\rightarrow_2$ *be an S-implication, then* (5) *is equivalent to the following*:

$$S(g_1(u,v), N(A^*(u))) \geq \alpha(u,v), \tag{14}$$

where $g_1(u,v) = S(N(R_1(u,v)), B^*(v))$.

Proof. For the S-implication $\rightarrow_2$, there exist a t-conorm S and a strong negation N such that $a \rightarrow_2 b = S(N(a), b)$. Then, we get

$$\begin{aligned} &R_1(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \\ &\quad = S(N(R_1(u,v)), S(N(A^*(u)), B^*(v))) \\ &\quad = S(S(N(R_1(u,v)), B^*(v)), N(A^*(u))) \\ &\quad = S(g_1(u,v), N(A^*(u))), \end{aligned}$$

where we let $g_1(u,v) = S(N(R_1(u,v)), B^*(v))$. Therefore, (5) is equivalent to (14). □

The following Proposition 5.2 provides a special $\alpha(u,v)$-MaxT-solution.

Proposition 5.2. *Let* $\rightarrow_2$ *be an S-implication, and* $g_1(u,v) \geq \alpha(u,v)$, *then the* $\alpha(u,v)$*-MaxT-solution is* $A^*(u) = 1, u \in U$.

Proof. There are a t-conorm S and a strong negation N such that $a \rightarrow_2 b = S(N(a), b)$. Noting that $g_1(u,v) \geq \alpha(u,v)$, we get from the definition of the t-conorm that

$$S(g_1(u,v), N(A^*(u))) \geq g_1(u,v) \geq \alpha(u,v).$$

So any fuzzy set A^* in $\langle F(U), \leq_F\rangle$ can let (14) and (5) hold, and then be an $\alpha(u,v)$-FMT-solution. Consequently, the maximum one (i.e., the $\alpha(u,v)$-MaxT-solution) is $A^*(u) = 1, u \in U$. □

The basic t-conorms are as follows ($a, b \in [0,1]$):

(i) Maximum: $S_M(a,b) = \max(a,b)$,
(ii) Probabilistic sum: $S_P(a,b) = a + b - ab$,
(iii) Lukasiewicz: $S_{LK}(a,b) = \min(a+b, 1)$,
(iv) Nilpotent maximum:

$$S_{nM}(a,b) = \begin{cases} 1, & a+b \geq 1, \\ \max(a,b), & \text{otherwise}, \end{cases}$$

(v) Drastic sum:

$$S_D(a,b) = \begin{cases} 1, & a, b \in (0,1], \\ \max(a,b), & \text{otherwise}. \end{cases}$$

The following Theorem 5.1 obtains the $\alpha(u,v)$-MaxT-solution for main t-conorms.

Theorem 5.1. *Let* $\rightarrow_2$ *be an S-implication* $I_{S,N}$, *and* $F_u = \{v \in V \mid g_1(u,v) < \alpha(u,v)\}$.

(i) *If* S *takes* S_M, *then the* $\alpha(u,v)$*-MaxT-solution is*

$$A^*(u) = \inf_{v\in F_u}\{N(\alpha(u,v))\}, u \in U.$$

(ii) *If* S *takes* S_P, *then the* $\alpha(u,v)$*-MaxT-solution is*

$$A^*(u) = \inf_{v\in F_u}\left\{N\left(\frac{\alpha(u,v) - g_1(u,v)}{1 - g_1(u,v)}\right)\right\}, \quad u \in U.$$

(iii) *If* S *takes* S_{LK}, *then the* $\alpha(u,v)$*-MaxT-solution is*

$$A^*(u) = \inf_{v\in F_u}\{N(\alpha(u,v) - g_1(u,v))\}, \quad u \in U.$$

(iv) *If* S *takes* S_{nM}, *then the* $\alpha(u,v)$*-MaxT-solution is*

$$A^*(u) = \inf_{v\in F_u}\{N(\alpha(u,v)) \vee N(1 - g_1(u,v))\}, \quad u \in U.$$

Proof. Here, we only prove the case of (iii) as an example, the remainders can be proved similarly. Note that N is a strong negation.

To begin with, we shall verify

$$A^*(u) = \inf_{v\in F_u}\{N(\alpha(u,v) - g_1(u,v))\}, \quad u \in U \tag{15}$$

is an $\alpha(u,v)$-FMT-solution, i.e., B^* can make (14) hold, which means that (5) holds.

If $y \notin F_u$, then $g_1(u, y) \geq \alpha(u, y)$, and

$$S_{LK}(g_1(u, y), N(A^*(u))) \geq \alpha(u, y),$$

i.e., (14) and (5) hold.

If $y \in F_u$, then $g_1(u, y) < \alpha(u, y)$, and thus it follows from (15) and the property of S_{LK} that

$$\begin{aligned} &S_{LK}(g_1(u, y), N(A^*(u))) \\ &\quad \geq S_{LK}(g_1(u, y), N(N(\alpha(u, y) - g_1(u, y)))) \\ &\quad = S_{LK}(g_1(u, y), \alpha(u, y) - g_1(u, y)) \\ &\quad = g_1(u, y) + \alpha(u, y) - g_1(u, y) \\ &\quad = \alpha(u, y). \end{aligned}$$

Then, (14) and (5) hold. Therefore, A^* expressed as (15) is an $\alpha(u, v)$-FMT-solution, i.e., $A^* \in \mathbb{F}_{\alpha(u,v)}$.

Furthermore, we shall verify that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. Suppose that $C \in \langle F(U), \leq_F \rangle$, and that

$$R_1(u, v) \rightarrow_2 (C(u) \rightarrow_2 B^*(v)) \geq \alpha(u, v)$$

holds for any $u \in U, v \in V$, then it follows from Proposition 5.1 that

$$S_{LK}(g_1(u, v), N(C(u))) \geq \alpha(u, v) \tag{16}$$

holds for any $u \in U, v \in V$.

If F_u is empty, then it follows from (15) that $A^*(u) = 1$, and thus $A^*(u) \geq C(u)$ $(u \in U)$.

If F_u is not empty, then for any $y \in F_u$, we have $g_1(u, y) < \alpha(u, y)$.

(a) If $g_1(u, y) + N(C(u)) \geq 1$, then

$$S_{LK}(g_1(u, y), N(C(u))) = 1.$$

Here,

$$N(C(u)) \geq 1 - g_1(u, y) \geq \alpha(u, y) - g_1(u, y)$$

and then

$$C(u) \leq N(\alpha(u, y) - g_1(u, y)).$$

Then, we get from (15) that

$$C(u) \leq A^*(u)(u \in U).$$

(b) If $g_1(u, y) + N(C(u)) < 1$, then it follows from (16) that

$$S_{LK}(g_1(u, y), N(C(u))) = g_1(u, y) + N(C(u)) \geq \alpha(u, y).$$

Hence,

$$N(C(u)) \geq \alpha(u, y) - g_1(u, y)$$

and then

$$C(u) \leq N(\alpha(u, y) - g_1(u, y)),$$

and thus $C(u) \leq A^*(u)$ $(u \in U)$.

Therefore, we have $C \leq_F A^*$. These imply that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$.

Together we obtain that A^* is the $\alpha(u, v)$-MaxT-solution in the light of Definition 3.2. □

The following Proposition 5.3 proves the existence of $\alpha(u, v)$-SupT-quasi solution.

Proposition 5.3. *For the $\alpha(u, v)$-FMT-differently implicational algorithm, there exists a unique fuzzy set C^* (in $\langle F(U), \leq_F\rangle$) such that*

(C1) *$C \leq_F C^*$ for any $C \in \mathbb{F}_{\alpha(u,v)}$, and*
(C2) *there is $D \in \mathbb{F}_{\alpha(u,v)}$ satisfying $D(u_0) > C^*(u_0) - \varepsilon$ for any $u_0 \in U$ and $\varepsilon > 0$, then C^* is the $\alpha(u, v)$-SupT-quasi solution.*

Proof. It follows from Lemma 2.2 that $\langle F(U), \leq_F\rangle$ is a complete lattice. Since the nonempty set $\mathbb{F}_{\alpha(u,v)} \subset F(U)$, we get that $C^* = \sup \mathbb{F}_{\alpha(u,v)}$ uniquely exists. We shall seek out the demanded C^*.

Because $\langle [0, 1], \leq\rangle$ is a complete lattice and $\{C(u_0)|\ C \in \mathbb{F}_{\alpha(u,v)}\} \subset [0, 1]$ $(u_0 \in U)$, it follows that

$$\sup\{C(u_0)|\ C \in \mathbb{F}_{\alpha(u,v)}\} \triangleq C_1(u_0)$$

uniquely exists. From the definition of supremum, we have that $C(u_0) \leq C_1(u_0)$ for any $C \in \mathbb{F}_{\alpha(u,v)}$ and that there exists $D \in \mathbb{F}_{\alpha(u,v)}$ such that

$$D(u_0) > C_1(u_0) - \varepsilon$$

for any $\varepsilon > 0$. Let u_0 respectively takes every element in U, and then we achieve the value of $C_1(u_0)$ for any $u_0 \in U$, as a result, there exists a fuzzy set C^* such that

$$C^*(u)|_{u=u_0} = C_1(u_0).$$

Consequently, C^* satisfies $(C1)$.

Moreover, we already know that there exists $D_1 \in \mathbb{F}_{\alpha(u,v)}$ satisfying

$$D_1(u_0) > C^*(u_0) - \varepsilon$$

for any $u_0 \in U$ and $\varepsilon > 0$. Let u_0 respectively takes every element in U, and then there is a fuzzy set D making

$$D(u)|_{u=u_0} = D_1(u_0)$$

hold, which is evidently what we demand. Therefore, $(C2)$ holds for C^*.

Since the partial order relation is according to the pointwise order, it is evident to find

$$C^* = \sup \mathbb{F}_{\alpha(u,v)},$$

and then C^* is the $\alpha(u, v)$-SupT-quasi solution. □

The following Proposition 5.4 gets the $\alpha(u, v)$-SupT-quasi solution for S_D.

Proposition 5.4. *If $\to_2$ is an S-implication $I_{S,N}$, and S takes S_D, then the $\alpha(u,v)$-SupT-quasi solution is as follows*:

$$A^*(u) = \inf_{v \in F_u} \{N(\alpha(u,v))\}, \quad u \in U, \tag{17}$$

where $F_u = \{v \in V \mid g_1(u,v) = 0\}$.

Proof. Note that N is a strong negation. We shall prove that A^* expressed as (17) satisfies $(C1)$ and $(C2)$ in Proposition 5.3.

(i) Suppose that C is any fuzzy set in $\mathbb{F}_{\alpha(u,v)}$. Then, C makes (14), i.e.,

$$S_D(g_1(u,v), N(C(u))) \geq \alpha(u,v) \tag{18}$$

hold for any $u \in U, v \in V$.

If $F_u \neq \emptyset$, then for any $y \in F_u$, we have $g_1(u,y) = 0$ and $A^*(u) = \inf_{v \in F_u} \{N(\alpha(u,v))\}$. Then, it follows from (18) that

$$\begin{aligned} S_D(g_1(u,y), N(C(u))) &= S_D(0, N(C(u))) \\ &= N(C(u)) \geq \alpha(u,y) \quad (u \in U). \end{aligned}$$

Then, $C(u) \leq N(\alpha(u,y))$, and thus $C(u)$ is a lower bound of $\{N(\alpha(u,v)) \mid v \in F_u\}$, and thus $C(u) \leq A^*(u)$ $(u \in U)$.
If $F_u = \emptyset$, then it follows from (17) that $A^*(u) = 1 \geq C(u)$ $(u \in U)$.
Together, we have $C \leq_F A^*$ for any $C \in \mathbb{F}_{\alpha(u,v)}$, i.e., A^* satisfies $(C1)$ in Proposition 5.3.

(ii) For any $\varepsilon > 0$, let

$$D(u) = \begin{cases} \inf_{v \in F_u} \{N(\alpha(u,v))\}, & F_u \neq \emptyset, \\ 1 - \varepsilon/\varepsilon_0, & F_u = \emptyset, \end{cases}$$

where ε_0 is the integer satisfying $\varepsilon + 1 < \varepsilon_0 \leq \varepsilon + 2$. Thus, $D(u_0) > A^*(u_0) - \varepsilon$ for any $u_0 \in U$.

We shall verify that $D(u_0)$ makes (14), i.e.,

$$S_D(g_1(u_0,v), N(D(u_0))) \geq \alpha(u_0,v) \tag{19}$$

hold for any $v \in V$.

If $F_{u_0} \neq \emptyset$, then $D(u_0) = \inf_{v \in F_{u_0}} \{N(\alpha(u_0,v))\} < 1$. It can be divided into two cases. (a) If $v \in F_{u_0}$, then $g_1(u_0,v) = 0$, and it follows from the property of S_D that

$$\begin{aligned} S_D(g_1(u_0,v), N(D(u_0))) &= S_D(0, N(D(u_0))) \\ &= N(D(u_0)) = N(\inf_{v \in F_{u_0}} \{N(\alpha(u_0,v))\}) \\ &\geq N(N(\alpha(u_0,v))) = \alpha(u_0,v), \end{aligned}$$

i.e., (19) holds. (b) If $v \notin F_{u_0}$, then $g_1(u_0,v) > 0$, and thus we get from the property of S_D that $S_D(g_1(u_0,v), N(D(u_0))) = 1 \geq \alpha(u_0,v)$. So, we also get that (19) holds.

If $F_{u_0} = \emptyset$, then $D(u_0) = 1 - \varepsilon/\varepsilon_0 < 1$ and $g_1(u_0, v) > 0$ for any $v \in V$, and it is similar to validate that (19) holds. Together, we get $D \in \mathbb{F}_{\alpha(u,v)}$, thus A^* satisfies $(C2)$ in Proposition 5.3.

Therefore, it follows from Proposition 5.3 that A^* expressed as (17) is the $\alpha(u, v)$-SupT-quasi solution. □

Remark 5.1. If $\rightarrow_2$ is an S-implication $I_{S,N}$, and S takes S_D, then the $\alpha(u, v)$-MaxT-solution (i.e., the maximum of $\mathbb{F}_{\alpha(u,v)}$) cannot ensure to exist. In fact, when there exists (u_0, v_0) such that $\alpha(u_0, v_0) > g_1(u_0, v_0) > 0$ and $F_{u_0} = \emptyset$, then $A^*(u_0) = 1$, and thus

$$\begin{aligned} S_D(g_1(u_0, v_0), N(A^*(u_0))) &= S_D(g_1(u_0, v_0), 0) \\ &= g_1(u_0, v_0) < \alpha(u_0, v_0), \end{aligned}$$

i.e., (14) does not hold, so the $\alpha(u, v)$-SupT-quasi solution $A^* \notin \mathbb{F}_{\alpha(u,v)}$, which results in that there does not exist the maximum of $\mathbb{F}_{\alpha(u,v)}$.

The following Lemma 5.1 shows some specific S-implications.

Lemma 5.1. *For the S-implication $I_{S,N}$, N takes N_s, then*:

(i) *If S takes S_M, then the S-implication is $I_{KD}(a, b) = (1 - a) \vee b$;*
(ii) *If S takes S_P, then the S-implication is $I_{RC}(a, b) = 1 - a + ab$;*
(iii) *If S takes S_{LK}, then the S-implication is I_L;*
(iv) *If S takes S_{nM}, then the S-implication is I_0;*
(v) *If S takes S_D, then the S-implication is* $I_{DP}(a, b) = \begin{cases} b, & a = 1, \\ a', & b = 0, \\ 1, & \text{otherwise.} \end{cases}$

It is easy to prove Proposition 5.5 (from Theorem 5.1 and Proposition 5.4). It shows specific $\alpha(u, v)$-MaxT-solutions for some S-implications.

Proposition 5.5. *Let the S-implication $\rightarrow_2 \in \{I_{KD}, I_{RC}, I_L, I_0, I_{DP}\}$, then*:

(i) *Let $\rightarrow_2$ take I_{KD}, then the $\alpha(u, v)$-MaxT-solution is*

$$A^*(u) = \inf_{v \in F_u} \{1 - \alpha(u, v)\}, \quad u \in U,$$

where $F_u = \{v \in V \mid (R_1(u, v))' \vee B^*(v) < \alpha(u, v)\}$.

(ii) *Let $\rightarrow_2$ take I_{RC}, then the $\alpha(u, v)$-MaxT-solution is*

$$A^*(u) = \inf_{v \in F_u} \left\{ \frac{1 - \alpha(u, v)}{R_1(u, v) - R_1(u, v) \times B^*(v)} \right\}, \quad u \in U,$$

where $F_u = \{v \in V \mid (R_1(u, v))' + R_1(u, v) \times B^*(v) < \alpha(u, v)\}$.

(iii) *Let $\rightarrow_2$ take I_L, then the $\alpha(u, v)$-MaxT-solution is the same as Proposition 4.3 (i).*

(iv) *Let $\rightarrow_2$ take I_0, then the $\alpha(u, v)$-MaxT-solution is the same as Proposition 4.3 (iv).*

(v) *Let* $\to_2$ *take* I_{DP}, *then the* $\alpha(u,v)$*-SupT-quasi solution is*

$$A^*(u) = \inf_{v \in F_u} \{1 - \alpha(u,v)\}, \quad u \in U,$$

where $F_u = \{v \in V \mid (R_1(u,v))' \vee B^*(v) = 0\}$.

The contents given above focus on the case of specific S-implications. Furthermore, we shall provide the unified form of the $\alpha(u,v)$-MaxT-solutions for S-implications from another viewpoint.

Theorem 5.2. *If* $\to_2$ *is an S-implication* $I_{S,N}$ *where* S *is right-continuous, then the* $\alpha(u,v)$*-MaxT-solution can be computed as follows* ($u \in U$):

$$A^*(u) = \inf_{v \in V} \{N(T(N(B^*(v)), T(R_1(u,v), \alpha(u,v))))\}, \tag{20}$$

in which T *is the mapping residual to* $\to_2$.

Proof. It follows from Propositions 2.1 and 2.2 that the residual pair $(T, \to_2)$ does exist. We shall verify A^* expressed as (20) makes (5) hold for any $u \in U, v \in V$. Indeed, it follows from (20) that ($u \in U, v \in V$)

$$A^*(u) \le N(T(N(B^*(v)), T(R_1(u,v), \alpha(u,v)))).$$

It follows from Proposition 2.2 that the S-implication $\to_2$ satisfies ($P14$) (w.r.t. the strong negation N), and $(T, \to_2)$ is a residual pair, so we have ($u \in U, v \in V$)

$$\begin{aligned} &T(N(B^*(v)), T(R_1(u,v), \alpha(u,v))) \le N(A^*(u)), \\ &T(R_1(u,v), \alpha(u,v)) \le N(B^*(v)) \to_2 N(A^*(u)), \\ &T(R_1(u,v), \alpha(u,v)) \le A^*(u) \to_2 B^*(v). \end{aligned}$$

Then, we obtain that (5) holds for any $u \in U, v \in V$. Thus, $A^* \in \mathbb{F}_{\alpha(u,v)}$.

Next, we shall verify that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. Suppose that $C \in \langle F(U), \le_F \rangle$, and that

$$R_1(u,v) \to_2 (C(u) \to_2 B^*(v)) \ge \alpha(u,v)$$

holds for any $u \in U, v \in V$. Considering that $(T, \to_2)$ is a residual pair and that $\to_2$ satisfies ($P14$) w.r.t. N, we obtain ($u \in U, v \in V$)

$$\begin{aligned} &T(R_1(u,v), \alpha(u,v)) \le C(u) \to_2 B^*(v), \\ &T(R_1(u,v), \alpha(u,v)) \le N(B^*(v)) \to_2 N(C(u)), \\ &T(N(B^*(v)), T(R_1(u,v), \alpha(u,v))) \le N(C(u)), \\ &C(u) \le N(T(N(B^*(v)), T(R_1(u,v), \alpha(u,v)))). \end{aligned}$$

Thus, $C(u)$ is a lower bound of ($u \in U$)

$$\{N(T(N(B^*(v)), T(R_1(u,v), \alpha(u,v)))) \mid v \in V\}.$$

Hence, it follows from (20) that $C \le_F A^*$. These imply that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$.

Together, we achieve that A^* is the $\alpha(u,v)$-MaxT-solution by virtue of Definition 3.2. □

Remark 5.2. Since S_M, S_P, S_{LK}, S_{nM} are right-continuous, we get from Theorem 5.2 that the $\alpha(u,v)$-MaxT-solution can be expressed as (20) for related S-implications, which can also obtain corresponding conclusions for I_{KD}, I_{RC}, I_L, I_0 in Proposition 5.5.

Proposition 5.6. *If* $\rightarrow_2$ *is an S-implication* $I_{S,N}$ *where* S *is right-continuous, and* $\rightarrow_2$ *is also an R-implication, then* $(u \in U, v \in V)$

$$N(T(N(B^*(v)), T(R_1(u,v), \alpha(u,v)))) = T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B^*(v), \tag{21}$$

where T *is the mapping residual to* $\rightarrow_2$.

Proof. Suppose that any $x \in [0,1]$. From the conditions (that $\rightarrow_2$ satisfies) together with the residuation condition, we know that the following formulas are equivalent to each other $(u \in U, v \in V)$:

$$\begin{aligned}
&x \leq N(T(N(B^*(v)), T(R_1(u,v), \alpha(u,v)))),\\
&T(N(B^*(v)), T(R_1(u,v), \alpha(u,v))) \leq N(x),\\
&T(R_1(u,v), \alpha(u,v)) \leq N(B^*(v)) \rightarrow_2 N(x),\\
&T(R_1(u,v), \alpha(u,v)) \leq x \rightarrow_2 B^*(v),\\
&x \leq T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B^*(v).
\end{aligned}$$

Because x is arbitrary, it follows that (21) holds. □

Remark 5.3. When $\rightarrow_2$ is an S-implication $I_{S,N}$ where S is right-continuous, and $\rightarrow_2$ is also an R-implication, it follows from Proposition 5.6 that the $\alpha(u,v)$-MaxT-solutions obtained from Theorems 4.2 and 5.2 are equivalent.

If there exist n rules instead of only one rule, then the FMT problem (2) should be transformed into:

$$\text{FMT : from } A_i \rightarrow B_i \quad \text{and} \quad B^*, \text{ calculate } A^*. \tag{22}$$

For such case, the general rule is frequently chosen to be

$$GR(u,v) \triangleq \vee_{i=1}^{n} (A_i(u) \rightarrow_1 B_i(v))$$

(see Refs. 13–15, 19). Therefore, (5) should be changed to:

$$GR(u,v) \rightarrow_2 (A^*(u) \rightarrow_2 B^*(v)) \geq \alpha(u,v). \tag{23}$$

Suppose that $\rightarrow_2$ employs an R-implication or S-implication, and that the $\alpha(u,v)$-MaxT-solution from (5) is $\psi(A(u) \rightarrow_1 B(v))$, then it is easy to find that the $\alpha(u,v)$-MaxT-solution derived from (23) is $\psi(GR(u,v))$.

6. Continuity of the $\alpha(u,v)$-FMT-Differently Implicational Algorithm

Suppose that d is a distance between fuzzy sets.

Definition 6.1. A fuzzy reasoning method for FMT (2) is a mapping $g: F(V) \to F(U)$.

(i) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(g(B_1^*), g(B_2^*)) < \varepsilon$ whenever $d(B_1^*, B_2^*) < \delta$ for any $B_1^*, B_2^* \in F(V)$, then g is said to be uniformly continuous in metric d;
(ii) For any $\varepsilon > 0$, if there exists $\delta > 0$ making $d(g(B^*), g(B)) < \varepsilon$ whenever $d(B^*, B) < \delta$ for any $B^* \in F(V)$, then g is said to be continuous at $B \in F(V)$ in metric d.

Because the practical problems frequently contain finite elements, we suppose that U, V are finite, i.e., $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_n\}$. Here we consider the two frequently used metrics, i.e., the uniform metric d_{UF} and Hamming metric d_{HM} as follows (where $A, B \in F(U)$):

$$d_{\mathrm{UF}}(A, B) = \sup_{u \in U} |A(u) - B(u)|,$$
$$d_{\mathrm{HM}}(A, B) = \frac{1}{m} \sum_{u \in U} |A(u) - B(u)|.$$

It is easy to prove Lemma 6.1.

Lemma 6.1. $|a \wedge c - b \wedge c| \leq |a - b|$, $|a \vee c - b \vee c| \leq |a - b|$, *where* $a, b, c \in [0, 1]$.

In Ref. 29, the following Lemma is obtained.

Lemma 6.2. *If* $U \to R$ *mappings* f, g *are bounded, in which* U *is a nonempty set and* R *is the set of real number, then for any* $u \in U$, *we have*

(i) $|\sup_{u \in U} f(u) - \sup_{u \in U} g(u)| \leq \sup_{u \in U} |f(u) - g(u)|$;
(ii) $|\inf_{u \in U} f(u) - \inf_{u \in U} g(u)| \leq \sup_{u \in U} |f(u) - g(u)|$.

It is easy to obtain Lemma 6.3.

Lemma 6.3. $d_{\mathrm{HM}}(A, B) \leq d_{\mathrm{UF}}(A, B)$, *in which* $A, B \in F(U)$.

Lemma 6.4. *If the* $\alpha(u, v)$*-FMT-differently implicational algorithm is uniformly continuous in* d_{UF}, *then it is also uniformly continuous in* d_{HM}.

Proof. Suppose that the $\alpha(u, v)$-FMT-differently implicational algorithm is uniformly continuous in d_{UF}. Then for any $\varepsilon > 0$, there exists $\delta > 0$ making $d_{\mathrm{UF}}(g(B_1^*), g(B_2^*)) < \varepsilon$ whenever $d_{UF}(B_1^*, B_2^*) < \delta$ for any $B_1^*, B_2^* \in F(V)$. It follows from Lemma 6.3 that $d_{\mathrm{HM}}(g(B_1^*), g(B_2^*)) \leq d_{\mathrm{UF}}(g(B_1^*), g(B_2^*)) < \varepsilon$. As a result, the $\alpha(u, v)$-FMT-differently implicational algorithm is also uniformly continuous in d_{HM}. □

Theorem 6.1. *Assume that the R-implication* $\to_2$ *satisfies*

($P15$) *I is continuous w.r.t. the second variable,*

then the $\alpha(u,v)$*-FMT-differently implicational algorithm is uniformly continuous in* d_{UF}*, and thus continuous in* d_{UF}.

Proof. Aiming at any inputs $B_1^*, B_2^* \in F(V)$, we shall verify the continuous property of the $\alpha(u,v)$-FMT-differently implicational algorithm. Note that $\to_2$ satisfies ($P15$), so $\to_2$ is uniformly continuous w.r.t. its second variable on [0,1]. As a result, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ making

$$|\,(T(R_1(u,v),\alpha(u,v)) \to_2 B_1^*(v)) - (T(R_1(u,v),\alpha(u,v)) \to_2 B_2^*(v))\,| < \varepsilon \qquad (24)$$

hold if $|\,B_1^*(v) - B_2^*(v)| < \delta_1$ $(v \in V)$.

It follows from Theorem 4.2 that the $\alpha(u,v)$-MaxT-solutions for B_1^*, B_2^* are as follows respectively:

$$A_1^*(u) = \inf_{v\in V}\{\,T(R_1(u,v),\alpha(u,v)) \to_2 B_1^*(v)\}, \quad u \in U,$$

$$A_2^*(u) = \inf_{v\in V}\{\,T(R_1(u,v),\alpha(u,v)) \to_2 B_2^*(v)\}, \quad u \in U.$$

We employ $\delta = \delta_1$. Suppose that

$$d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta.$$

Then, $\sup_{v\in V}|B_1^*(v) - B_2^*(v)| < \delta$ and $|B_1^*(v) - B_2^*(v)| < \delta = \delta_1$ $(v \in V)$. So (24) holds, and according to Lemmas 6.1 and 6.2, we get

$$\begin{aligned} &d_{UF}(A_1^*, A_2^*) \\ &\quad = \sup_{u\in U}|A_1^*(u) - A_2^*(u)| \\ &\quad = \sup_{u\in U}|\inf_{v\in V}\{\,T(R_1(u,v),\alpha(u,v)) \to_2 B_1^*(v)\} \\ &\qquad - \inf_{v\in V}\{\,T(R_1(u,v),\alpha(u,v)) \to_2 B_2^*(v)\}| \\ &\quad \le \sup_{u\in U}\sup_{v\in V}|(\,T(R_1(u,v),\alpha(u,v)) \to_2 B_1^*(v)) \\ &\qquad - (\,T(R_1(u,v),\alpha(u,v)) \to_2 B_2^*(v))| \\ &\quad < \sup_{u\in U}\sup_{v\in V}\varepsilon = \varepsilon. \end{aligned}$$

That is, there exists $\delta > 0$ such that $d_{\mathrm{UF}}(A_1^*, A_2^*) < \varepsilon$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta$, therefore the $\alpha(u,v)$-FMT-differently implicational algorithm is uniformly continuous in d_{UF}. And thus it is also continuous in d_{UF}. □

It follows from Theorem 6.1 and Lemma 6.4 that we can obtain Theorem 6.2.

Theorem 6.2. *Assume that the R-implication* $\to_2$ *satisfies* ($P15$), *then the* $\alpha(u,v)$*-FMT-differently implicational algorithm is uniformly continuous in* d_{HM}*, and thus continuous in* d_{HM}.

Theorems 6.1 and 6.2 verifies the continuity of the $\alpha(u,v)$-FMT-differently implicational algorithm for the R-implication satisfying $(P15)$.

For $\rightarrow_2 \in \{I_L, I_{Go}, I_{ep}, I_{y-0.5}\}$, $\rightarrow_2$ satisfies $(P15)$. So, we can get Proposition 6.1. It provides the continuous result for some specific R-implications.

Proposition 6.1. *If $\rightarrow_2 \in \{I_L, I_{Go}, I_{ep}, I_{y-0.5}\}$, then the $\alpha(u,v)$-FMT-differently implicational algorithm is uniformly continuous in $d \in \{d_{\mathrm{UF}}, d_{\mathrm{HM}}\}$, and thus continuous in $d \in \{d_{\mathrm{UF}}, d_{\mathrm{HM}}\}$.*

Moreover, when $\rightarrow_2$ is only right-continuous w.r.t. the second variable, whether is the $\alpha(u,v)$-FMT-differently implicational algorithm continuous? Here, we shall analyze the typical case of $\rightarrow_2 \in \{I_0, I_G\}$.

Proposition 6.2. *For any $B_1^*, B_2^* \in F(V)$, there exists $\delta_0 > 0$ such that if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_0$, then $F_{1u} = F_{2u}$ $(u \in U)$, in which $F_{1u} = \{v \in V \mid T(R_1(u,v), \alpha(u,v)) > B_1^*(v)\}$, $F_{2u} = \{v \in V \mid T(R_1(u,v), \alpha(u,v)) > B_2^*(v)\}$.*

Proof. For any $u \in U$, we shall analyze the relationship between F_{1u} and F_{2u}. We employ

$$\delta_1 = \min_{v \in F_{1u}} [T(R_1(u,v), \alpha(u,v)) - B_1^*(v)].$$

Obviously, we have

$$T(R_1(u,v), \alpha(u,v)) - B_1^*(v) \geq \delta_1 > 0 \ (v \in V). \tag{25}$$

Suppose that $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_1$. Then, $\sup_{v \in V} |B_1^*(v) - B_2^*(v)| < \delta_1$, and thus we get

$$B_1^*(v) - \delta_1 < B_2^*(v) < B_1^*(v) + \delta_1 (v \in V). \tag{26}$$

For any $v_0 \in F_{1u}$, we get $T(R_1(u,v_0), \alpha(u,v_0)) - B_1^*(v_0) > 0$, and it follows from (25), (26) that

$$\begin{aligned} B_2^*(v_0) &< B_1^*(v_0) + \delta_1 \leq B_1^*(v_0) + [T(R_1(u,v_0), \alpha(u,v_0)) - B_1^*(v_0)] \\ &= T(R_1(u,v_0), \alpha(u,v_0)), \end{aligned}$$

thus, $v_0 \in F_{2u}$, hence, we get $F_{1u} \subset F_{2u}$.

Take

$$\delta_2 = \min_{v \in F_{2u}} [T(R_1(u,v), \alpha(u,v)) - B_2^*(v)].$$

Similarly, we can obtain $F_{2u} \subset F_{1u}$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_2$.

Choose

$$\delta_0 = \min\{\delta_1, \delta_2\},$$

thus $F_{1u} \subset F_{2u}$ and $F_{2u} \subset F_{1u}$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_0$. Consequently, we achieve that if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_0$ then $F_{1u} = F_{2u}$. □

Theorem 6.3. *If $\rightarrow_2 \in \{I_0, I_G\}$, then the $\alpha(u,v)$-FMT-differently implicational algorithm is uniformly continuous in d_{UF}, and thus continuous in d_{UF}.*

Proof. We only prove the case of I_0, while the case of I_G can be similarly verified.

Suppose that $\rightarrow_2$ takes I_0. It follows from Proposition 6.2 that there exists $\delta_0 > 0$ such that $F_{1u} = F_{2u}$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_0$ $(u \in U)$. For any $\varepsilon > 0$, take

$$\delta = \min\{\delta_0, \varepsilon\}.$$

Suppose that $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta$. Then, we have

$$F_{1u} = F_{2u} \quad (u \in U),$$

and $|B_1^*(v) - B_2^*(v)| < \delta$ $(v \in V)$. Thus, it follows from Theorem 4.2, Lemmas 6.1 and 6.2 that if $F_{1u} \neq \emptyset$, then

$$\begin{aligned}
&d_{\mathrm{UF}}(A_1^*, A_2^*) \\
&\quad = \sup_{u\in U} |A_1^*(u) - A_2^*(u)| \\
&\quad = \sup_{u\in U} |\inf_{v\in V}\{T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B_1^*(v)\} \\
&\qquad - \inf_{v\in V}\{T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B_2^*(v)\}| \\
&\quad = \sup_{u\in U} |\inf_{v\in F_{1u}}\{T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B_1^*(v)\} \\
&\qquad - \inf_{v\in F_{2u}}\{T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B_2^*(v)\}| \\
&\quad = \sup_{u\in U} |\inf_{v\in F_{1u}}\{T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B_1^*(v)\} \\
&\qquad - \inf_{v\in F_{1u}}\{T(R_1(u,v), \alpha(u,v)) \rightarrow_2 B_2^*(v)\}| \\
&\quad = \sup_{u\in U} |\inf_{v\in F_{1u}}\{[1 - (R_1(u,v), \alpha(u,v))] \vee B_1^*(v)\} \\
&\qquad - \inf_{v\in F_{1u}}\{[1 - (R_1(u,v), \alpha(u,v))] \vee B_2^*(v)\}| \\
&\quad \leq \sup_{u\in U} \inf_{v\in F_{1u}} |\{[1 - (R_1(u,v), \alpha(u,v))] \vee B_1^*(v)\} \\
&\qquad - \{[1 - (R_1(u,v), \alpha(u,v))] \vee B_2^*(v)\}| \\
&\quad \leq \sup_{u\in U} \inf_{v\in F_{1u}} |B_1^*(v) - B_2^*(v)| \\
&\quad < \sup_{u\in U} \inf_{v\in F_{1u}} \delta \\
&\quad = \delta \\
&\quad \leq \varepsilon.
\end{aligned}$$

If $F_{1u} = \emptyset$, then we get

$$d_{\mathrm{UF}}(A_1^*, A_2^*) = \sup_{u\in U} |\,1 - 1\,| = 0 < \varepsilon.$$

That is, there always exists $\delta > 0$ such that $d_{\mathrm{UF}}(A_1^*, A_2^*) < \varepsilon$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta$, therefore the $\alpha(u,v)$-FMT-differently implicational algorithm is uniformly continuous in d_{UF}, and then it is continuous in d_{UF}. □

We can get Theorem 6.4 from Theorem 6.3 and Lemma 6.4.

Theorem 6.4. *If* $\to_2 \in \{I_0, I_G\}$, *then the* $\alpha(u,v)$*-FMT-differently implicational algorithm is uniformly continuous in* d_{HM}, *and thus continuous in* d_{HM}.

Proposition 6.2, Theorems 6.3 and 6.4 prove the continuity of the $\alpha(u,v)$-FMT-differently implicational algorithm in which $\to_2 \in \{I_0, I_G\}$.

Proposition 6.3. *For any* $B_1^*, B_2^* \in F(V)$, *there exists* $\delta_0 > 0$ *such that if* $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_0$, *then* $G_{1u} = G_{2u}$ $(u \in U)$, *in which* $G_{1u} = \{v \in V \mid (R_1(u,v))' \vee B_1^*(v) < \alpha(u,v)\}$, $G_{2u} = \{v \in V \mid (R_1(u,v))' \vee B_2^*(v) < \alpha(u,v)\}$.

Proof. Denote

$$\begin{aligned} G_{1u}^* &= \{v \in V \mid B_1^*(v) < \alpha(u,v)\}, \\ G_{2u}^* &= \{v \in V \mid B_2^*(v) < \alpha(u,v)\}, \\ G_u^{**} &= \{v \in V \mid (R_1(u,v))' < \alpha(u,v)\}. \end{aligned}$$

Then, we get

$$G_{1u} = G_{1u}^* \cup G_u^{**}, \tag{27}$$

$$G_{2u} = G_{2u}^* \cup G_u^{**}. \tag{28}$$

We take

$$\delta_1 = \min_{v \in G_{1u}^*} [\alpha(u,v) - B_1^*(v)].$$

Obviously, we have

$$\alpha(u,v) - B_1^*(v) \geq \delta_1 > 0 \ (v \in V). \tag{29}$$

Suppose that $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_1$. Then, $\sup_{v \in V} |B_1^*(v) - B_2^*(v)| < \delta_1$, and thus we get

$$B_1^*(v) - \delta_1 < B_2^*(v) < B_1^*(v) + \delta_1 (v \in V). \tag{30}$$

For any $v_0 \in G_{1u}^*$, we get $\alpha(u,v_0) - B_1^*(v_0) > 0$, and it follows from (29), (30) that

$$B_2^*(v_0) < B_1^*(v_0) + \delta_1 \leq B_1^*(v_0) + [\alpha(u,v_0) - B_1^*(v_0)] = \alpha(u,v_0),$$

thus $v_0 \in G_{2u}^*$, hence, we get $G_{1u}^* \subset G_{2u}^*$.

Take

$$\delta_2 = \min_{v \in G_{2u}^*} [\alpha(u,v) - B_2^*(v)].$$

Similarly, we can obtain $G_{2u}^* \subset G_{1u}^*$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_2$.

Choose

$$\delta_0 = \min\{\delta_1, \delta_2\}.$$

Therefore, it follows from (27) and (28) that if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_0$, then $G_{1u}^* = G_{2u}^*$ and thus $G_{1u} = G_{2u}$. □

Theorem 6.5. *If $\rightarrow_2$ takes I_{KD}, then the $\alpha(u,v)$-FMT-differently implicational algorithm is uniformly continuous in d_{UF}, and thus continuous in d_{UF}.*

Proof. Suppose that $\rightarrow_2$ employs I_{KD}. We get from Proposition 6.3 that there exists $\delta_0 > 0$ making $G_{1u} = G_{2u}$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta_0$ $(u \in U)$. For any $\varepsilon > 0$, choose

$$\delta = \min\{\delta_0, \varepsilon\}.$$

Suppose that $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta$. Then, we get

$$G_{1u} = G_{2u} \quad (u \in U),$$

and $|B_1^*(v) - B_2^*(v)| < \delta$ $(v \in V)$. So we get from Proposition 5.5, Lemmas 6.1 and 6.2 that if $G_{1u} \neq \emptyset$ then

$$\begin{aligned} &d_{\mathrm{UF}}(A_1^*, A_2^*) \\ &\quad = \sup_{u\in U} |A_1^*(u) - A_2^*(u)| \\ &\quad = \sup_{u\in U} |\inf_{v\in V}\{1-\alpha(u,v)\} - \inf_{v\in V}\{1-\alpha(u,v)\}| \\ &\quad = \sup_{u\in U} |\inf_{v\in G_{1u}}\{1-\alpha(u,v)\} - \inf_{v\in G_{2u}}\{1-\alpha(u,v)\}| \\ &\quad = \sup_{u\in U} |\inf_{v\in G_{1u}}\{1-\alpha(u,v)\} - \inf_{v\in G_{1u}}\{1-\alpha(u,v)\}| \\ &\quad \leq \sup_{u\in U} \inf_{v\in G_{1u}} |[1-\alpha(u,v)] - [1-\alpha(u,v)]| \\ &\quad = \sup_{u\in U} \inf_{v\in G_{1u}} 0 \\ &\quad = 0 \\ &\quad < \varepsilon. \end{aligned}$$

If $G_{1u} = \emptyset$, then we obtain

$$d_{\mathrm{UF}}(A_1^*, A_2^*) = \sup_{u\in U} |\,1-1\,| = 0 < \varepsilon.$$

Consequently, there always exists $\delta > 0$ letting $d_{\mathrm{UF}}(A_1^*, A_2^*) < \varepsilon$ if $d_{\mathrm{UF}}(B_1^*, B_2^*) < \delta$, therefore the $\alpha(u,v)$-FMT-differently implicational algorithm is uniformly continuous in d_{UF}, and thus it is continuous in d_{UF}. □

From Theorem 6.5 and Lemma 6.4, we can prove Theorem 6.6.

Theorem 6.6. *If $\rightarrow_2$ takes I_{KD}, then the $\alpha(u,v)$-FMT-differently implicational algorithm is uniformly continuous in d_{HM}, and thus continuous in d_{HM}.*

Proposition 6.3, Theorems 6.5 and 6.6 verify the continuity of the $\alpha(u,v)$-FMT-differently implicational algorithm where $\rightarrow_2$ takes I_{KD}.

Remark 6.1. For all R-implications, the $\alpha(u,v)$-FMT-differently implicational algorithm (where $\rightarrow_2 \in \{I_L, I_G, I_{Go}, I_0, I_{ep}, I_{y-0.5}\}$) is uniformly continuous and continuous in $d_{\mathrm{UF}}, d_{\mathrm{HM}}$. In five typical S-implications, the $\alpha(u,v)$-FMT-differently implicational algorithm where $\rightarrow_2 \in \{I_0, I_L, I_{KD}\}$ is uniformly continuous and continuous in $d_{\mathrm{UF}}, d_{\mathrm{HM}}$. When $\to_2 \in \{I_{RC}, I_{DP}\}$, corresponding continuity cannot be assured. To sum up, the continuity of the $\alpha(u,v)$-FMT-differently implicational algorithm seems excellent.

7. Examples

Here, we shall provide two specific examples (including a continuous case and a discrete one) to deal with $\alpha(u,v)$-FMT-differently implicational algorithm, where I_L, I_0 are R-implications and also S-implications and I_{RC} is an S-implication.

Example 7.1. Let $U = V = [0,1]$, $A(u) = (u+1)/3$, $B(v) = 1-v$, $B^*(v) = 2/3$ and $\alpha(u,v) = (6+v-u)/9$ (in which $u, v \in [0,1]$). Suppose that $\to_2 = I_L$, $\to_1 = I_0$ in the $\alpha(u,v)$-FMT-differently implicational algorithm. We now calculate the $\alpha(u,v)$-MaxT-solution from Theorem 4.2 (which is more universal than Proposition 4.3(i)).

$$R_1(u,v) = A(u) \to_1 B(v) = \begin{cases} \dfrac{2-u}{3} \vee (1-v), & \text{if } u+3v > 2, \\ 1, & \text{if } u+3v \leq 2. \end{cases}$$

It follows from Theorem 4.2 that the $\alpha(u,v)$-MaxT-solution is as follows ($u \in U$):

$$\begin{aligned} A^*(u) &= \inf_{v \in V}\{T_L(R_1(u,v), \alpha(u,v)) \to_2 B^*(v)\} \\ &= \inf\left\{T_L\left(\frac{2-u}{3} \vee (1-v), \frac{6+v-u}{9}\right) \to_2 \frac{2}{3} \,\middle|\, v \in [0,1], u+3v > 2\right\} \\ &\quad \wedge \inf\left\{T_L\left(1, \frac{6+v-u}{9}\right) \to_2 \frac{2}{3} \,\middle|\, v \in [0,1], u+3v \leq 2\right\}. \end{aligned}$$

Obviously, $u+3v > 2$ implies $3v > 2-u \geq 1$ and

$$T_L\left(\frac{2-u}{3} \vee (1-v), \frac{6+v-u}{9}\right) \leq \frac{2-u}{3} \vee (1-v) \leq \frac{2}{3},$$

$$T_L\left(\frac{2-u}{3} \vee (1-v), \frac{6+v-u}{9}\right) \to_2 \frac{2}{3} = 1.$$

Thus, we obtain that

$$\begin{aligned} A^*(u) &= \inf\left\{T_L\left(1, \frac{6+v-u}{9}\right) \to_2 \frac{2}{3} \,\middle|\, v \in [0,1], u+3v \leq 2\right\} \\ &= \inf\left\{\frac{6+v-u}{9} \to_2 \frac{2}{3} \,\middle|\, v \in [0,1], u+3v \leq 2\right\} \\ &= \inf\left\{1 - \frac{6+v-u}{9} + \frac{2}{3} \,\middle|\, v \in [0,1], u+3v \leq 2, \frac{6+v-u}{9} > \frac{2}{3}\right\} \end{aligned}$$

$$= \inf\left\{\frac{9-v+u}{9}\;\middle|\; v\in[0,1], \frac{2-u}{3}\ge v>u\right\}.$$

We have two cases to be considered:

(a) Suppose that $(2-u)/3 > u$, i.e., $u < 1/2$, thus

$$\{v\in[0,1],(2-u)/3\ge v>u\}\neq\emptyset,$$

taking into account that $(9-v+u)/9$ is nonincreasing w.r.t v, we have

$$A^*(u)=\frac{9-\frac{2-u}{3}+u}{9}=\frac{25+4u}{27}.$$

(b) Suppose that $(2-u)/3\le u$, i.e., $u\ge 1/2$, thus

$$\{v\in[0,1],\ (2-u)/3\ge v>u\}=\emptyset,$$

we get $A^*(u)=\inf\emptyset=1$.

Together, we obtain that the $\alpha(u,v)$-MaxT-solution is

$$A^*(u)=\begin{cases}\dfrac{25+4u}{27}, & u<\dfrac{1}{2},\\[2ex] 1, & u\ge\dfrac{1}{2}.\end{cases}$$

Example 7.2. Let $\alpha(u,v)=(u^2-v)/2$, and $U=\{u_1\}$ where $u_1=1.2$, and $V=\{v_1,v_2,v_3,v_4\}$ where $v_1=0.0, v_2=0.2, v_3=0.4, v_4=0.6$. The rules and inputs are as follows:

$$\begin{aligned}
A_1&=\frac{0.3}{u_1}, & B_1&=\frac{0.8}{v_1}+\frac{0.4}{v_2}+\frac{1.0}{v_3}+\frac{0.6}{v_4},\\
A_2&=\frac{0.3}{u_1}, & B_2&=\frac{0.0}{v_1}+\frac{0.7}{v_2}+\frac{0.3}{v_3}+\frac{0.5}{v_4},\\
A_3&=\frac{0.6}{u_1}, & B_3&=\frac{0.7}{v_1}+\frac{0.3}{v_2}+\frac{0.9}{v_3}+\frac{0.2}{v_4},\\
A_4&=\frac{0.9}{u_1}, & B_4&=\frac{0.2}{v_1}+\frac{0.5}{v_2}+\frac{0.8}{v_3}+\frac{0.7}{v_4},\\
A_5&=\frac{0.9}{u_1}, & B_5&=\frac{0.5}{v_1}+\frac{0.9}{v_2}+\frac{0.5}{v_3}+\frac{0.0}{v_4},\\
B^*&=\frac{0.3}{v_1}+\frac{0.4}{v_2}+\frac{0.5}{v_3}+\frac{0.8}{v_4}.
\end{aligned}$$

This is an example for fuzzy classification based on fuzzy expert system, in which three classes respectively correpond to $A(u_1)=0.3, A(u_1)=0.6, A(u_1)=0.9$. Suppose $\to_2\,=I_L,\ \to_1\,=I_{RC}$ in the variable differently implicational algorithm for FMT. From Proposition 4.3, we can obtain the $\alpha(u,v)$-MaxT-solution is

$$A^*(u)=\inf_{v\in F_u}\{2-GR(u,v)-\alpha(u,v)+B^*(v)\},$$

where

$$F_u = \{v \in V \mid GR(u,v) + \alpha(u,v) - 1 > B^*(v)\}.$$

We have:

$$\begin{aligned} GR(u_1, v_1) &= \vee_{i=1}^{5}(A_i(u_1) \to_1 B_i(v_1)) \\ &= (0.3 \to_1 0.8) \vee (0.3 \to_1 0.0) \vee (0.6 \to_1 0.7) \\ &\quad \vee (0.9 \to_1 0.2) \vee (0.9 \to_1 0.5) \\ &= 0.94 \vee 0.7 \vee 0.82 \vee 0.28 \vee 0.55 = 0.94. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} GR(u_1, v_2) &= 0.82 \vee 0.91 \vee 0.58 \vee 0.55 \vee 0.91 = 0.91. \\ GR(u_1, v_3) &= 1.0 \vee 0.79 \vee 0.94 \vee 0.82 \vee 0.55 = 1.0. \\ GR(u_1, v_4) &= 0.88 \vee 0.85 \vee 0.52 \vee 0.73 \vee 0.1 = 0.88. \end{aligned}$$

For $u_1 = 1.2$, it is easy to find $F_{u_1} = \{v_1, v_2, v_3\}$. As a result, we obtain the $\alpha(u,v)$-MaxT-solution as follows:

$$\begin{aligned} A^*(u_1) &= \inf_{v \in F_{u_1}} \{2 - GR(u_1, v) - \alpha(u_1, v) + B^*(v)\}, \\ &= [2 - GR(u_1, v_1) - \alpha(u_1, v_1) + B^*(v_1)] \vee \dots \\ &\quad \vee [2 - GR(u_1, v_3) - \alpha(u_1, v_3) + B^*(v_3)] \\ &= [2 - 0.94 - \alpha(1.2, 0.0) + 0.3] \\ &\quad \vee [2 - 0.91 - \alpha(1.2, 0.2) + 0.4] \\ &\quad \vee [2 - 1.0 - \alpha(1.2, 0.4) + 0.5] \\ &= 0.64 \vee 0.87 \vee 0.98 = 0.98. \end{aligned}$$

Because 0.98 is nearest to 0.9, the third class is what we need.

8. Conclusions

In this paper, the variable differently implicational algorithm is investigated for the FMT problem. First of all, fundamental properties of the variable differently implicational algorithm for FMT are researched. Especially, new differently implicational principle for FMT is proposed, which improves the previous one in Ref. 15. Then, unified forms of variable differently implicational algorithm are achieved for FMT, in which $\to_2$ respectively employs the R-implication and S-implication. Furthermore, the optimal solutions of variable differently implicational algorithm for FMT are provided for six R-implications and five S-implications. After that, as for the important index of continuity, this algorithm is uniformly continuous and continuous for all R-implications and most of S-implication, and then its continuity seems excellent. Finally, two specific computing examples (including a continuous case and a discrete one) of variable differently implicational algorithm for FMT are shown.

Furthermore, some other properties of the variable differently implicational algorithm (e.g., robustness, stability[30,31]), and how to reasonably design corresponding fuzzy system,[32,33] and how to apply it to decision making,[34–36] will be our next work emphases.

Acknowledgments

This research has been partially supported by the JSPS KAKENHI (No. 15H01712), the National Natural Science Foundation of China (No. 61432004), the Natural Science Foundation of Anhui Province (Nos. 1408085MKL15 and 1508085QF129), and the China Postdoctoral Science Foundation (Nos. 2014T70585 and 2012M521218).

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