International Journal of Information Technology & Decision Making
Vol. 16, No. 2 (2017) 443–471

DOI: 10.1142/S0219622014500746

Fuzzy Systems Based on Universal Triple I Method and Their Response Functions

Yiming Tang

Information and Communication Engineering
Postdoctoral Research Station
Hefei University of Technology
Heifei 230009, China
and
AnHui Province Key Laboratory of Affective
Computing and Advanced Intelligent Machine
School of Computer and Information
Hefei University of Technology
Hefei 230009, China
tym608@163.com

Fuji Ren*

Institute of Technology and Science
The University of Tokushima
Minami Josanjima, Tokushima 770-8506, Japan
and
AnHui Province Key Laboratory of Affective Computing
and Advanced Intelligent Machine
School of Computer and Information
Hefei University of Technology
Hefei 230009, China
ren@is.tokushima-u.ac.jp

Published 18 September 2014

The fuzzy systems based on the universal triple I method are investigated, and then their response functions are analyzed. First, the conclusions show that 100 fuzzy systems via the universal triple I method are approximately interpolation functions, which can be used in practical systems, and that 90 ones are approximately fitted functions, which may be usable. Second, as its special cases, the Compositional Rule of Inference (CRI) method and the triple I method are discussed, with the results that 19 fuzzy systems via the CRI method and 2 ones via the triple I method are practicable. Therefore, the universal triple I method has larger effective choosing space, which can obtain more usable fuzzy systems than the others. Lastly, it is found that the first implication and second implication, respectively, embody the function of rule base and reasoning mechanism, further demonstrating the reasonability of the universal triple I method.

Keywords: Fuzzy system; fuzzy reasoning; response function; triple I method; CRI method.

*Corresponding author.

1. Introduction

Nowadays fuzzy reasoning and fuzzy system play an important role in fuzzy control, decision making, artificial intelligence, image processing, natural language processing and affective computing.[1–4] Every fuzzy system based on the CRI method[5–7] can be regarded as an interpolation method,[8,9] which is a kind of approximation to a certain response function. Currently, a fuzzy system used in the practical system is usually constructed via the CRI method and a certain implication operator (defined as a mapping $[0,1]^2 \to [0,1]$).

To improve the CRI method, Wang put forward the triple I method of fuzzy reasoning in 1999.[10] The main idea of the triple I method is as follows:

For known $A \in F(X)$, $B \in F(Y)$, and $A^* \in F(X)$ (or $B^* \in F(Y)$), seeks the smallest $B^* \in F(Y)$ (or the largest $A^* \in F(X)$) such that

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \tag{1}$$

takes the maximum for any $x \in X, y \in Y$, where $F(X), F(Y)$ are respectively the sets of all fuzzy subsets on the input universe X and output universe Y, while $\to$ is an implication operator.

Following that, many scholars carried through a series of researches related to the triple I method, including the triple I method,[11,12] the α-triple I method,[13,14] the restriction theory of triple I method,[15,16] the reverse triple I method,[17,18] reversibility property[10,13] and logic basis of related triple I method[19,20] and so on. Such results show that the triple I method possesses many advantages such as its logic basis, excellent reversibility property, and the property of pointwise optimization, thus the triple I method is better than the CRI method from the viewpoint of logic.

On the other hand, from the viewpoint of fuzzy system (which includes fuzzier, fuzzy reasoning method, and defuzzier), the triple I method is further analyzed. In general, if a fuzzy system is only of step response ability (but not universal approximator[21]), then it can hardly be utilized in any practical systems. Therefore, it is vitally significant to analyze the response function of a fuzzy system. The fuzzy systems employing singleton fuzzier, centroid defuzzier and the triple I method or CRI method for fuzzy reasoning have been discussed. In Ref. 22, two fuzzy systems can be used in 51 fuzzy systems based on the triple I method, while in Refs. 23, 12 fuzzy systems can be utilized in 23 fuzzy systems based on the CRI method. By direct comparisons, the fuzzy systems via the triple I method are inferior to the ones via the CRI method.[24] Based on implication operator R_p with parameter p, the fuzzy systems via the triple I method are of step response ability and then not practicable.[25] In Ref. 26, two fuzzy systems are practicable in 11 fuzzy systems via the triple I method, while four fuzzy systems are usable in 11 ones via the CRI method, which are constructed by the same 11 implication operators.

Therefore, there are very few usable fuzzy systems based on the triple I method; and the triple I method is not as good as the CRI method from the viewpoint of fuzzy system, weakening the value of triple I method as an improvement of the CRI method. Such inferior response ability and practicability will hold back the further development of the triple I method to a great extent.

To solve this problem, an important way is presented to improve the triple I method. The CRI method can be seen as a special case of the triple I method only if three implication operators in (1) are different.[24] In detail, the CRI method (which is expressed as $B^*(y) = \sup_{x\in X}\{A^*(x) \wedge (A(x) \to B(y))\}, y \in Y$) can be regarded as the triple I method where formula (1) is changed into

$$(A(x) \to B(y)){\to_2}(A^*(x){\to_2}B^*(y)),$$

where $\to_2$ takes the Mamdani operator R_M.

Enlightened by this idea, we can let the latter two implication operators be same and the first one unlimited, that is, generalize (1) to:

$$(A(x){\to_1}B(y)){\to_2}(A^*(x){\to_2}B^*(y)), \tag{2}$$

where $\to_1$ and $\to_2$ (respectively called the first implication and second implication in the sequel) can take different implication operators, and the triple I method derived from (2) is called the differently implicational universal triple I method of (1, 2, 2) type (the universal triple I method for short). In Ref. 27, we have already proposed and discussed the universal triple I method with some preliminary results. In this paper, we shall systematically investigate the fuzzy systems based on the universal triple I method and their response functions.

The rest of this paper is organized as follows. In Sec. 2, some definitions and results of implication operators together with residual pairs are recalled; moreover, for the solutions of universal triple I method, the related conclusions are introduced. In Secs. 3 and 4, the single-input single-output (SISO) and double-input single-output (DISO) fuzzy systems via the universal triple I algorithm are constructed respectively, and then the response functions of corresponding fuzzy systems are obtained. Section 5 draws the conclusion.

2. Preliminaries

2.1. *Some related implication operators*

From Ref. 28 the definition of residual pair is shown as Definition 2.1, which can help establish unified forms of the universal triple I method.

Definition 2.1. Let $\to$ and $\otimes$ be two $[0,1]^2 \to [0,1]$ mappings, $(\to, \otimes)$ is called a residual pair or, $\to$ and $\otimes$ are residual to each other, if

$$a \otimes b \leq c \quad \text{iff} \;\; b \leq a \to c$$

holds for any $a, b, c \in [0,1]$, in which iff denotes "if and only if".

From Ref. 27, (C1)–(C3) are the conditions for an implication operator to construct residual pair (see the following theorem).

Theorem 2.1. *Let* $\to: [0,1]^2 \to [0,1]$ *be an implication operator satisfying*

(C1) $a \to b$ *is nondecreasing w.r.t.* b $(a, b \in [0,1])$,
(C2) $a \to b$ *is right-continuous w.r.t.* b $(a \in [0,1],\ b \in [0,1))$,
(C3) $\{y \in [0,1] | a \to y = 1\} \neq \emptyset$ $(a \in [0,1])$,

and define $\otimes_\to : [0,1]^2 \to [0,1]$ *as follows*

$$a \otimes_\to b = \wedge\{y \in [0,1] | b \leq a \to y\}, \quad a, b \in [0,1],$$

then $(\to, \otimes_\to)$ *is a residual pair, and* $a \to b = \vee\{y \in [0,1]|\ a \otimes_\to y \leq b\}$(*where* $\wedge, \vee$ *denotes infimum and supremum, respectively*).

In this paper, we mainly consider 10 familiar implication operators. They are Mamdani operator R_M, Zadeh operator R_Z, Lukasiewicz operator R_L, Gödel operator R_G, Goguen operator R_{Go}, R_0 operator (from Refs. 10 and 29), and R_{ep}, $R_{dp-\beta}$ $(\beta \in [0,1])$, $R_{y-0.5}$ (from Refs. 27 and 30), together with revised Reichenbach operator R_{10} (from Ref. 31) as the following (where x' denotes $1-x$).

$$R_M(a,b) = a \wedge b, \quad R_Z(a,b) = a' \vee (a \wedge b),$$

$$R_L(a,b) = \begin{cases} 1, & a \leq b, \\ a' + b, & a > b, \end{cases} \quad R_G(a,b) = \begin{cases} 1, & a \leq b, \\ b, & a > b, \end{cases}$$

$$R_{Go}(a,b) = \begin{cases} 1, & a = 0, \\ (b/a) \wedge 1, & a \neq 0, \end{cases} \quad R_0(a,b) = \begin{cases} 1, & a \leq b, \\ a' \vee b, & a > b, \end{cases}$$

$$R_{ep}(a,b) = \begin{cases} 1, & a \leq b, \\ (2b - ab)/(a + b - ab), & a > b, \end{cases}$$

$$R_{dp-\beta}(a,b) = \begin{cases} 1, & a \leq b, \\ b\beta/a, & \beta \geq a > b, \\ b, & a > b,\ a > \beta, \end{cases} \quad (\beta \in (0,1)),$$

$$R_{y-0.5}(a,b) = \begin{cases} 1, & a \leq b, \\ 1 - (\sqrt{1-b} - \sqrt{1-a})^2, & a > b, \end{cases}$$

$$R_{10}(a,b) = \begin{cases} 1, & a \leq b, \\ a' + ab, & a > b. \end{cases}$$

2.2. *The solutions of universal triple I method*

In Ref. 27, we have already accomplished some works of the universal triple I method of fuzzy reasoning, and here we briefly sketch in some related definitions and results (from Ref. 27) in this subsection.

Definition 2.2. Suppose that $A, A^* \in F(X)$, $B \in F(Y)$, and that nonempty set $\mathbb{E}$ is the set of B^* which makes (2) get its maximum for any $x \in X, y \in Y$ in $< F(Y), \leq_F >$, and finally that D^* is the infimum of $\mathbb{E}$. If D^* is the minimum of $\mathbb{E}$, then D^* is called a MinP-solution.

Remark 2.1. Definition 2.2 gives the definition of optimal solution of the universal triple I method. It is noted that $< F(Y), \leq_F >$ is a complete lattice where the partial order relation $\leq_F$ on $F(Y)$ is defined as: $A_1 \leq_F A_2$ iff $A_1(y) \leq A_2(y)$ for $\forall y \in Y$ (where $A_1, A_2 \in F(Y)$).

Theorem 2.2. *Suppose that* $\rightarrow_2$ *is an implication operator satisfying* $(C1)$–$(C4)$ $a \leq b$ *iff* $a \rightarrow b = 1(a, b \in [0,1])$, *and that* $\otimes$ *is its residual mapping, then the MinP-solution can be expressed as*

$$B^*(y) = \sup_{x\in X}\{A^*(x) \otimes (A(x) \rightarrow_1 B(y))\}, \quad y \in Y.$$

From Theorem 2.2, when the implication operator $\rightarrow_2$ (in the universal triple I method) satisfies (C1)–(C4), then the residual pair $(\rightarrow_2, \otimes)$ can be generated, and unified form of the universal triple I method can be established to allow different implication operators to be employed in the same manner.

In the implication operators mentioned above, $R_L, R_G, R_{Go}, R_0, R_{ep}, R_{dp-\beta}, R_{y-0.5}$, R_{10} satisfy (C1)–(C4), thus Theorem 2.2 holds for these implication operators.

Proposition 2.1. *The operations corresponding to* R_G,R_{Go},R_L,R_0,R_{10},R_{ep}, $R_{dp-\beta}$,$R_{y-0.5}$ *in residual pairs are as follows, respectively.*

$$a \otimes_G b = a \wedge b, \quad a \otimes_{Go} b = a \times b, \quad a \otimes_L b = \begin{cases} a+b-1, & a+b>1, \\ 0, & a+b \leq 1, \end{cases}$$

$$a \otimes_0 b = \begin{cases} a \wedge b, & a+b>1, \\ 0, & a+b \leq 1, \end{cases} \quad a \otimes_{10} b = \begin{cases} [(a+b-1)/a] \wedge a, & a+b>1, \\ 0, & a+b \leq 1, \end{cases}$$

$$a \otimes_{ep} b = ab/[2-(a+b-ab)], \quad a \otimes_{dp-\beta} b = ab/\max(a,b,\beta)(\beta \in [0,1]),$$

$$a \otimes_{y-0.5} b = \begin{cases} 1-(g(a,b))^2, & g(a,b) \leq 1, \\ 0, & g(a,b) > 1, \end{cases} \quad (g(a,b) = \sqrt{1-a} + \sqrt{1-b}).$$

Proposition 2.2. (i) *If* $\rightarrow_2$ *is* R_Z, *then the MinP-solution can be expressed as*

$$B^*(y) = \sup_{x\in E_y}\{A^*(x) \wedge (A(x) \rightarrow_1 B(y))\} \quad (y \in Y),$$

where $E_y = \{x \in X \mid (A^*(x))' \vee 0.5 < (A(x) \rightarrow_1 B(y))\}$.

(ii) *If* $\rightarrow_2$ *is* R_M, *then the MinP-solution can be expressed as*

$$B^*(y) = \sup_{x\in X}\{A^*(x) \wedge (A(x) \rightarrow_1 B(y))\}, \quad y \in Y. \tag{3}$$

Remark 2.2. Notice that the solution of the CRI method is computed as

$$B^*(y) = \sup_{x\in X}\{A^*(x) \wedge (A(x) \rightarrow B(y))\} \quad (y \in Y), \tag{4}$$

where $\rightarrow$ is an implication operator.[32,33] Thus, from Proposition 2.2(ii), it is easy to know that when $\rightarrow_2$ is R_M (and $\rightarrow_1$ in formula (3) takes $\rightarrow$ in (4)), the universal triple I method degenerates into the CRI method.

Remark 2.3. When $\rightarrow_1 = \rightarrow_2$ in (2), it is obvious that the universal triple I method degenerates into the triple I method.

The fuzzy partition is an important structure for fuzzy system, where its definition is as follows.

Definition 2.3. Let Z be any nonempty set and $\mathbb{C} = \{C_i\}_{(1\leq i\leq n)}$ a family of normal fuzzy sets on Z, where the peak-point of C_i is z_i (i.e., the unique point satisfying $C_i(z_i) = 1$ in Z). $\mathbb{C}$ is called a fuzzy partition of Z if $(\forall z \in Z)(\sum_{i=1}^{n} C_i(z) = 1)$ holds, and C_i is defined as a base element in $\mathbb{C}$. Thus $\mathbb{C}$ is also called a group of base elements of Z.

Remark 2.4. It is obvious that Definition 2.3 implies $(\forall i, j)(i \neq j \Rightarrow z_i \neq z_j)$ and that $\mathbb{C}$ has Kronecker property (i.e., $C_i(z_j) = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$).

3. SISO Fuzzy Systems Based on the Universal Triple I Method and their Response Functions

Based on the universal triple I method, in this section we shall construct the corresponding SISO fuzzy systems, and then analyze their response functions.

3.1. *Construction of the SISO fuzzy systems via the universal triple I method*

Here we shall establish the SISO fuzzy systems via the universal triple I method.

Let X and Y be the input universe and output universe, respectively. Denote

$$\mathbb{A} = \{A_i\}_{(1\leq i\leq n)}, \quad \mathbb{B} = \{B_i\}_{(1\leq i\leq n)},$$

where $A_i \in F(X)$, $B_i \in F(Y)$ in which $F(X), F(Y)$ are the sets of all fuzzy subsets on X, Y, respectively. $\mathbb{A}, \mathbb{B}$ are regarded as linguistic variables, thus the fuzzy reasoning rules can be expressed as follows:

$$\text{If } x \text{ is } A_i, \text{ then } y \text{ is } B_i, \; i = 1, \ldots, n, \tag{5}$$

where $x \in X$, $y \in Y$ are called base variables.

Similar to Refs. 8, 22 and 23, the reasoning relation of the ith inference rule can be regarded as a fuzzy relation from X to Y $(i = 1, \ldots, n)$, denoting by $A_i(x)\rightarrow_1 B_i(y)$ (where $\rightarrow_1$ is an implication operator). And such n rules can be connected by "OR" relation, thus the whole reasoning rule should be

$$\rho_1(x, y) \triangleq \vee_{i=1}^{n} (A_i(x)\rightarrow_1 B_i(y)).$$

Given $A^* \in F(X)$, the reasoning conclusion $B^* \in F(Y)$ can be gotten by the universal triple I method of fuzzy reasoning. Therefore, formula (2) should be turned into the following formula:

$$\rho_1(x, y)\rightarrow_2(A^*(x)\rightarrow_2 B^*(y)). \tag{6}$$

We have given some MinP-solutions derived from (2) (e.g., $\rightarrow_2 \in \{R_M, R_Z\}$), then it is similar to get the MinP-solutions derived from (6). We only analyze the case $\rightarrow_2 = R_0$ as an example. For (2), by Theorem 2.2 we can easily get that the MinP-solution is

$$B^*(y) = \sup_{x \in E_y} \{A^*(x) \wedge (A(x) \rightarrow_1 B(y))\},$$

where $E_y = \{x \in X | (A^*(x))' < A(x) \rightarrow_1 B(y)\}$. If we compare (2) with (6), it is easy to find that $(A(x) \rightarrow_1 B(y))$ is displaced by $\rho_1(x, y)$, which is the unique difference. Thus, the solving process is not changed basically, we can similarly achieve that the MinP-solution from (6) is

$$B^*(y) = \sup_{x \in E_y} \{A^*(x) \wedge \rho_1(x, y)\},$$

where $E_y = \{x \in X | (A^*(x))' < \rho_1(x, y)\}$.

Now we shall construct the SISO fuzzy system via the universal triple I method as the following three steps:

(i) Since the input value of a fuzzy system should be a crisp number $x^* \in X$, to use the universal triple I method of fuzzy reasoning, x^* should be changed into a fuzzy set by defining a singleton[8,22,23]

$$A^*(x) = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases} \triangleq A^*_{x^*},$$

which is called fuzzier.

(ii) Then, from A^*, we get the reasoning conclusion B^* by the universal triple I method.

(iii) Because B^* is a fuzzy set, it should be transformed into a crisp value y^* as the output value, which is called defuzzier. Usually the centroid defuzzier[22–26] is in common use, that is

$$y^* = \frac{\int_Y yB^*(y)\,dy}{\int_Y B^*(y)\,dy}.$$

However, the centroid defuzzier makes no sense when $B^*(y) \equiv 0$. In Ref. 30, several defuzziers, including the centroid defuzzier, the center average defuzzier and the defuzzier of average from the maximum were provided. The last one (that is, the defuzzier of average from the maximum) which employs

$$hgt(B^*) = \left\{ y \in Y | B^*(y) = \sup_{y \in Y} B^*(y) \right\}, \quad y^* = \frac{\int_{hgt(Y)} y\,dy}{\int_{hgt(Y)} dy},$$

(noting $hgt(Y) = Y$ here), is partly similar to the centroid defuzzier. As a result, we mainly utilize the centroid defuzzier; and when $B^*(y) \equiv 0$, we adopt the defuzzier of average from the maximum. This method (taking two defuzziers) has been proved to be valid.[26,27]

To sum up, there is an output $y^* = F(x^*)$ for each input x^*. Thus, the SISO fuzzy system via the universal triple I method is constructed.

In order to investigate response functions of fuzzy systems, suppose that $\mathbb{A} = \{A_i\}_{(1\le i\le n)}$ and $\mathbb{B} = \{B_i\}_{(1\le i\le n)}$ are respectively fuzzy partitions of X and Y (in which A_i, B_i are integrable functions); moreover, assume that X and Y are all real number intervals, e.g., $X = [a, b]$ and $Y = [c, d]$ where $a < x_1 < x_2 < \cdots < x_n < b$ and $c < y_1 < y_2 < \cdots < y_n < d$, in which x_i, y_i are respectively peak-points of A_i, B_i and corresponding distributions (for $\{a, x_1, x_2, \ldots, x_n, b\}$, $\{c, y_1, y_2, \ldots, y_n, d\}$) are basically uniform.

Let $h_1 = y_1 - c$, $h_i = y_i - y_{i-1}$ $(i = 2, 3, \ldots, n)$ and $h = \max_{1\le i\le n}\{h_i\}$. Since $\mathbb{A}$ and $\mathbb{B}$ are all fuzzy partitions, they have Kronecker property: $A_i(x_j) = \delta_{ij} = B_i(y_j)$. By the definition of definite integral, we achieve for the centroid defuzzier:

$$y^* = \frac{\int_Y yB^*(y)\,dy}{\int_Y B^*(y)\,dy} \approx \frac{\sum_{i=1}^n y_i B^*(y_i)h_i}{\sum_{i=1}^n B^*(y_i)h_i}. \tag{7}$$

Similarly, we get for the defuzzier of average from the maximum:

$$y^* = \frac{\int_{hgt(Y)} ydy}{\int_{hgt(Y)} dy} \approx \frac{\sum_{i=1}^n y_i h_i}{\sum_{i=1}^n h_i} \triangleq c_0.$$

3.2. *Response functions of SISO fuzzy systems via the universal triple I method*

For the constructed SISO fuzzy systems via the universal triple I method, here we shall investigate their response functions.

For the MinP-solution B^*, its equivalent form in the SISO fuzzy system is going to be investigated (see Theorem 3.1).

Theorem 3.1. (i) *Let* $\to_2 \in \{R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, *then the MinP-solution* $B^*(y) = \rho_1(x^*, y)$ *in a SISO fuzzy system via the universal triple I method.*

(ii) *Let* $\to_2 \in \{R_0, R_L, R_Z, R_{10}\}$, *then the MinP-solution* $B^*(y) = \rho_1(x^*, y)$ *when* $x^* \in E_y$ *and* $B^*(y) = 0$ *when* $x^* \in X - E_y$ *in a SISO fuzzy system via the universal triple I method.*

Proof. (i) Suppose $\to_2 \in \{R_G, R_{Go}, R_M, R_{ep}\}$. We only prove the case of R_G as an example. It follows from Theorem 2.2 that the MinP-solution can be expressed as

$$B^*(y) = \sup_{x\in X}\{A^*(x) \wedge \rho_1(x, y)\},$$

where $\rho_1(x, y) = \vee_{i=1}^n (A_i(x)\to_1 B_i(y))$. As for input x^*, we get a singleton $A^*_{x^*} = \begin{cases} 1, & x = x^* \\ 0, & x \ne x^* \end{cases}$. Thus it is evident to get $B^*(y) = \rho_1(x^*, y)$.

Suppose $\to_2 \in \{R_{y-0.5}, R_{dp-\beta}\}$, We only prove the case of $R_{y-0.5}$ as an example. It is similar to get the MinP-solution

$$\begin{aligned} B^*(y) &= \sup_{x\in X}\{A^*(x)\otimes_{y-0.5}\rho_1(x, y)\} \\ &= \sup_{x\in E_y}\{1 - (\sqrt{1 - A^*(x)} + \sqrt{1 - \rho_1(x, y)})^2\}, \quad y \in Y, \end{aligned}$$

where $E_y = \{x \in X | \sqrt{1 - A^*(x)} + \sqrt{1 - \rho_1(x, y)} \le 1\}$, $\rho_1(x, y) = \vee_{i=1}^n (A_i(x) \to_1 B_i(y))$. As for input x^*, we get a singleton $A^*_{x^*}$. It is easy to get $x^* \in E_y$. Note that $\otimes_{y-0.5}$ is a t-norm, then $1 \otimes_{y-0.5} b = b$ and $0 \otimes_{y-0.5} b = 0$ hold for any $b \in [0, 1]$. Thus

$$A^*(x) \otimes_{y-0.5} \rho_1(x, y) = 1 \otimes_{y-0.5} \rho_1(x^*, y) = \rho_1(x^*, y)$$

when $x = x^*$; and

$$A^*(x) \otimes_{y-0.5} \rho_1(x, y) = 0 \otimes_{y-0.5} \rho_1(x, y) = 0$$

when $x \in X - \{x^*\}$. Hence $B^*(y) = \sup_{x \in X}\{A^*(x) \otimes_{y-0.5} \rho_1(x, y)\} = \rho_1(x^*, y)$.

(ii) We only prove the case of $\to_2 = R_0$ as an example. It follows from Theorem 2.2 that the MinP-solution can be expressed as

$$B^*(y) = \sup_{x \in E_y}\{A^*(x) \wedge \rho_1(x, y)\},$$

where $E_y = \{x \in X | (A^*(x))' < \rho_1(x, y)\}$ and $\rho_1(x, y) = \vee_{i=1}^n (A_i(x) \to_1 B_i(y))$. As for input x^*, we get a singleton $A^*_{x^*}$. If $x^* \in E_y$, then we have $E_y = \{x^*\}$ by the structure of E_y, and thus $B^*(y) = \rho_1(x^*, y)$. If $x^* \in X - E_y$, then $E_y = \emptyset$;, and thus $B^*(y) = 0$. □

Based on the conditions where $\to_1$ satisfies as well as the characteristics of fuzzy partition (according to Definition 2.3 and Remark 2.4), we can prove Lemma 3.1, which provides the basis for the following theorems. In fact, to analyze the response function of fuzzy system via the universal triple I method, we find that (C5)–(C8) in Lemma 3.1 are respectively important conditions for $\to_1$ to determine the response function to a large extent.

Lemma 3.1. *In a SISO fuzzy system via the universal triple I method,*

(i) *if* $\to_1$ *satisfies one of the following conditions* $(a \in [0, 1])$:

(C5) $a \to 1 = 1$ *or* $a \to 0 = 1$,
(C6) $a \to 1 = (1 + a)/2$ *and* $a \to 0 \le 1/2$, *or* $a \to 0 = (1 + a)/2$ *and* $a \to 1 \le 1/2$, *then* $\rho_1(x^*, y_j) > 0$ *for any* $x^* \in X (j = 1, \ldots, n)$;

(ii) *if* $\to_1 = R_Z$, *then* $\rho_1(x^*, y_j) > 0$ *for any* $x^* \in X (j = 1, \ldots, n)$;
(iii) *if* $\to_1$ *satisfies one of the following conditions* $(a \in [0, 1])$

(C7) $a \to 1 = a$ *and* $a \to 0 = 0$,
(C8) $a \to 1 = 0$ *and* $a \to 0 \in \{a, 1 - a\}$,
then there exists y_j *such that* $\rho_1(x^*, y_j) > 0$ *for any* $x^* \in X$ $(j \in \{1, \ldots, n\})$.

Theorems 3.2–3.4 provide the response functions of SISO fuzzy systems via the universal triple I method, which are derived from the equivalent form of MinP-solution and the condition that $\to_1$ satisfies.

Theorem 3.2. *Suppose that the MinP-solution is* $B^*(y) = \rho_1(x^*, y)$, *and that* $\to_1$ *satisfies* (*C*7) *in a SISO fuzzy system via the universal triple I method. Then there exists a group of base functions* $\mathbb{A}^* = \{A^*_i\}_{(1 \le i \le n)}$ *such that the SISO fuzzy system is* *approximately a univariate piecewise interpolation function taking A_i^* as its base functions (i.e., $F(x) = \sum_{i=1}^{n} A_i^*(x)y_i$), and $\mathbb{A}^*$ is a fuzzy partition on X. Especially, if $\{y_i\}_{(1\le i\le n)}$ is an equidistant partition, then $\mathbb{A}^*$ degenerates into $\mathbb{A}$(i.e., $F(x) = \sum_{i=1}^{n} A_i(x)y_i$).*

Proof. As for $B^*(y) = \rho_1(x^*, y) = \vee_{i=1}^{n}(A_i(x^*)\to_1 B_i(y))$, since $\to_1$ satisfies (C7) and $B_k(y_i) = \delta_{ki}$, it follows from (7) that we obtain:

$$\begin{aligned} y^* &= \frac{\int_Y yB^*(y)\,dy}{\int_Y B^*(y)\,dy} \approx \frac{\sum_{i=1}^{n} y_i B^*(y_i)h_i}{\sum_{i=1}^{n} B^*(y_i)h_i} \\ &= \frac{\sum_{i=1}^{n} h_i[\vee_{k=1}^{n}(A_k(x^*)\to_1 B_k(y_i))]y_i}{\sum_{i=1}^{n} h_i[\vee_{k=1}^{n}(A_k(x^*)\to_1 B_k(y_i))]} = \frac{\sum_{i=1}^{n} h_i A_i(x^*)y_i}{\sum_{i=1}^{n} h_i A_i(x^*)}, \end{aligned} \tag{8}$$

where there exists y_i such that $B^*(y_i) = \rho_1(x^*, y_i) > 0$ ($i \in \{1,\ldots,n\}$) by Lemma 3.1 (iii), so $\sum_{i=1}^{n} B^*(y_i)h_i > 0$ and then (8) makes sense.

Denote

$$A_i^*(x^*) \triangleq h_i A_i(x^*) \bigg/ \left[\sum_{i=1}^{n} h_i A_i(x^*)\right],$$

then we have $y^* \approx \sum_{i=1}^{n} A_i^*(x^*)y_i$. Let $\mathbb{A}^* \triangleq \{A_i^*\}_{(1\le i\le n)}$, $F(x) \triangleq \sum_{i=1}^{n} A_i^*(x)y_i$. Considering $A_k(x_i) = \delta_{ki}$, it follows that

$$F(x_i) = \sum_{k=1}^{n} A_k^*(x_i)y_k = \left[\sum_{k=1}^{n} h_k A_k(x_i)y_k\right] \bigg/ \left[\sum_{k=1}^{n} h_k A_k(x_i)\right] = y_i$$

for $i = 1,\ldots,n$, then $F(x)$ is a univariate piecewise interpolation function which regards A_i^* as its base functions.

Furthermore,

$$\sum_{i=1}^{n} A_i^*(x) = \sum_{i=1}^{n}\left[h_i A_i(x) \bigg/ \left(\sum_{i=1}^{n} h_i A_i(x)\right)\right] = 1$$

holds for any $x \in X$, so $\mathbb{A}^*$ is a fuzzy partition on X.

At last, if $\{y_i\}_{(1\le i\le n)}$ is an equidistant partition (i.e., $(\forall i)(h_i = h)$), then it is evident that $A_i^* = A_i$, $\mathbb{A}^* = \mathbb{A}$, and hence $F(x) = \sum_{i=1}^{n} A_i(x)y_i$. □

Remark 3.1. In Theorem 3.2 (and also what follows), there exists an important word "approximately" for several times. Here we shall interpret the meaning of "approximately". For the case of centroid defuzzier, it is easy to find that the key of "approximately" lies in (see (7) and (8))

$$\frac{\int_Y yB^*(y)\,dy}{\int_Y B^*(y)\,dy} \approx \frac{\sum_{i=1}^{n} y_i B^*(y_i)h_i}{\sum_{i=1}^{n} B^*(y_i)h_i}.$$

It follows from the definition of definite integral that we can get the following interpretations (noting that the distribution for $\{c, y_1, y_2, \ldots, y_n, d\}$ is basically uniform): (i) if n is larger, then $\sum_{i=1}^{n} y_i B^*(y_i)h_i / \sum_{i=1}^{n} B^*(y_i)h_i$ is more approximate to $\int_Y yB^*(y)dy/\int_Y B^*(y)dy$; (ii) for any $\varepsilon > 0$, there exists a natural number N such that for any $n > N$ we have

$$\left|\frac{\int_Y yB^*(y)dy}{\int_Y B^*(y)dy} - \frac{\sum_{i=1}^n y_i B^*(y_i)h_i}{\sum_{i=1}^n B^*(y_i)h_i}\right| < \varepsilon.$$

It is noted that n is determined by users according to the actual demand. For the case of defuzzier of average from the maximum, we can get similar interpretations.

Theorem 3.3. *Suppose that the MinP-solution is* $B^*(y) = \rho_1(x^*, y)$, *and that* $\rightarrow_1$ *satisfies* $(C6)$ *or* $(C8)$, *or* $\rightarrow_1 = R_Z$ *in a SISO fuzzy system via the universal triple I method. Then there exists a group of base functions* $\mathbb{A}^* = \{A_i^*\}_{(1\le i\le n)}$ *such that the SISO fuzzy system is approximately a univariate piecewise fitted function regarding* A_i^* *as its base functions* (*i.e.*, $F(x) = \sum_{i=1}^n A_i^*(x)y_i$).

Proof. We only prove the case that $\rightarrow_1$ satisfies $a \rightarrow 1 = (1+a)/2$ and $a \rightarrow 0 \le 1/2$ as an example. As for $B^*(y) = \rho_1(x^*, y) = \vee_{i=1}^n (A_i(x^*)\rightarrow_1 B_i(y))$, since $B_k(y_i) = \delta_{ki}$, it follows from (7) that we obtain:

$$\begin{aligned} y^* &\approx \frac{\sum_{i=1}^n y_i B^*(y_i)h_i}{\sum_{i=1}^n B^*(y_i)h_i} = \frac{\sum_{i=1}^n h_i[\vee_{k=1}^n (A_k(x^*)\rightarrow_1 B_k(y_i))]y_i}{\sum_{i=1}^n h_i[\vee_{k=1}^n (A_k(x^*)\rightarrow_1 B_k(y_i))]} \\ &= \frac{\sum_{i=1}^n h_i[(1+A_i(x^*))/2]y_i}{\sum_{i=1}^n h_i[(1+A_i(x^*))/2]}, \end{aligned} \tag{9}$$

where $B^*(y_i) = \rho_1(x^*, y_i) > 0$ ($i = 1, \ldots, n$) from Lemma 3.1(i), so $\sum_{i=1}^n B^*(y_i)h_i > 0$ and then (9) makes sense.

Denote

$$C_i(x^*) \triangleq (1 + A_i(x^*))/2, \quad A_i^*(x^*) \triangleq h_i C_i(x^*) \bigg/ \left[\sum_{i=1}^n h_i C_i(x^*)\right],$$

thus $y^* \approx \sum_{i=1}^n A_i^*(x^*)y_i$. Let $\mathbb{A}^* \triangleq \{A_i^*\}_{(1\le i\le n)}$, $F(x) \triangleq \sum_{i=1}^n A_i^*(x)y_i$. Considering $A_j(x_i) = \delta_{ji}$, we get ($i = 1, \ldots, n$):

$$F(x_i) = \frac{\sum_{j=1}^n h_j[(1+A_j(x_i))/2]y_j}{\sum_{j=1}^n h_j[(1+A_j(x_i))/2]} = \frac{\sum_{j=1}^n h_j(1+A_j(x_i))y_j}{\sum_{j=1}^n h_j(1+A_j(x_i))} = \frac{h_i y_i + \sum_{j=1}^n h_j y_j}{h_i + \sum_{j=1}^n h_j}.$$

Obviously, it cannot make $F(x_i) = y_i$ always hold for every i, thus $F(x)$ is a univariate piecewise fitted function which regards A_i^* as its base functions. $\square$

Theorem 3.4. (i) *Suppose that the MinP-solution is* $B^*(y) = \rho_1(x^*, y)$, *and that* $\rightarrow_1$ *satisfies* $(C5)$ *in a SISO fuzzy system via the universal triple I method. Then the SISO fuzzy system is approximately a step response function* (*i.e.*, $F(x) = c_0$).

(ii) *Suppose that the MinP-solution is* $B^*(y) = a(a \in [0,1])$ *in a SISO fuzzy system via the universal triple I method, then the SISO fuzzy system is approximately a step response function* (*i.e.*, $F(x) = c_0$).

Proof. (i) As for $B^*(y) = \rho_1(x^*, y) = \vee_{i=1}^n (A_i(x^*) \to_1 B_i(y))$, since $\to_1$ satisfies (C5) and $B_k(y_i) = \delta_{ki}$, it follows from (7) that we have:

$$y^* \approx \frac{\sum_{i=1}^n y_i B^*(y_i) h_i}{\sum_{i=1}^n B^*(y_i) h_i} = \frac{\sum_{i=1}^n h_i [\vee_{k=1}^n (A_k(x^*) \to_1 B_k(y_i))] y_i}{\sum_{i=1}^n h_i [\vee_{k=1}^n (A_k(x^*) \to_1 B_k(y_i))]} = \frac{\sum_{i=1}^n h_i y_i}{\sum_{i=1}^n h_i} = c_0, \tag{10}$$

where $B^*(y_i) = \rho_1(x^*, y_i) > 0$ $(i = 1, \ldots, n)$ from Lemma 3.1(i), thus $\sum_{i=1}^n B^*(y_i) h_i > 0$ and then (10) makes sense. Thus the response function $F(x) = c_0$.

(ii) It can be divided into two cases.

(a) Suppose $a = 0$. Then $B^*(y) = 0$ and the centroid defuzzier makes on sense, so we utilize the defuzzier of average from the maximum. Thus $y^* \approx c_0$ and the response function can be expressed as $F(x) = c_0$.

(b) Suppose $a > 0$, considering $B_k(y_i) = \delta_{ki}$, it follows from (7) that we can easily get $y^* \approx \sum_{i=1}^n y_i h_i / \sum_{i=1}^n h_i = c_0$.

Therefore, the response function can be expressed as $F(x) = c_0$. □

Remark 3.2. The previous researches on response functions of fuzzy systems,[22,23,25–27] are commonly derived from some specific implication operators. However, Theorems 3.2–3.4 in this paper, are from the equivalent form of MinP-solutions in fuzzy systems, which provide a new research idea. By such a new idea, it is easier for us to grasp the essence of response functions.

If $\to_2$ employs specific implication operator in the SISO fuzzy systems via the universal triple I method, then we can get Corollaries 3.1–3.3. Here Corollaries 3.1 and 3.2 can be proved by virtue of Theorems 3.1–3.4. Then it follows from Corollaries 3.1 and 3.2 that Corollary 3.3 can be obtained.

Corollary 3.1. *Suppose that* $\to_2 \in \{R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ *in a SISO fuzzy system via the universal triple I method.*

(i) *Let* $\to_1$ *satisfy* $(C5)$, *then the SISO fuzzy system is approximately a step response function.*
(ii) *Let* $\to_1$ *satisfy* $(C7)$, *then the conclusion is the same as Theorem* 3.2.
(iii) *Let* $\to_1$ *satisfy* $(C6)$ *or* $(C8)$, *or* $\to_1 = R_Z$, *then the conclusion is the same as Theorem* 3.3.

Corollary 3.2. *Suppose that* $\to_2 \in \{R_0, R_L, R_Z, R_{10}\}$ *in a SISO fuzzy system via the universal triple I method.*

(i) *Let* $\to_1$ *satisfy* $(C5)$, *then the SISO fuzzy system is approximately a step response function.*
(ii) *Let* $\to_1$ *satisfy* $(C7)$, *then there are two cases to be considered:* (a) *Suppose* $x^* \in E_y$, *then the conclusion is the same as Theorem* 3.2. (b) *Suppose* $x^* \in X - E_y$, *then the SISO fuzzy system is approximately a step response function.*

(iii) *Let* $\to_1$ *satisfy* $(C6)$ *or* $(C8)$, *or* $\to_1 = R_Z$, *then there are two cases to be considered*: (a) *Suppose* $x^* \in E_y$, *then the conclusion is the same as Theorem* 3.3. (b) *Suppose* $x^* \in X - E_y$, *then the SISO fuzzy system is approximately a step response function.*

Corollary 3.3. *If* $\to_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ *and* $\to_1$ *satisfies* $(C5')$ $a \to 1 = 1 (a \in [0,1])$, *then the SISO fuzzy system via the universal triple I method is approximately a step response function* (*i.e.*, $F(x) = c_0$).

Remark 3.3. In Ref. 27, the following implication operators are also investigated.

$$R_{KD}(a,b) = a' \vee b, \quad R_{La}(a,b) = a \times b, \quad R_Y(a,b) = b^a,$$

$$R_R(a,b) = a' + ab, \quad R_{GR}(a,b) = \begin{cases} 1, & a \le b, \\ 0, & a > b, \end{cases}$$

$$R_{16}(a,b) = \begin{cases} 1, & a \le b, \\ a'/b', & a > b, \end{cases} \quad R_{17}(a,b) = \begin{cases} 1, & a \le b, \\ a', & a > b. \end{cases}$$

Moreover, from Theorem 6.3 of Ref. 27, when $\to_2 \in \{R_{KD}, R_{La}, R_Y, R_R, R_{GR}, R_{16}, R_{17}\}$, the SISO fuzzy system via the universal triple I method is approximately a step response function, which can hardly be used in practical systems. Therefore, in Secs. 3 and 4, we do not consider these implication operators as the second implication $\to_2$ (in the universal triple I method).

Remark 3.4. From Theorems 6.1 and 6.2 of Ref. 27, the response functions of the SISO fuzzy systems via the universal triple I method were discussed. It is easy to get that Corollaries 3.1(ii)(iv), 3.2(ii)(iv), and 3.3 in this paper, include the conclusions of Theorems 6.1 and 6.2 of Ref. 27. What is more, these corollaries are induced by Theorems 3.2–3.4, thus the related conclusions in this paper are superior to the ones in Ref. 27.

Remark 3.5. $R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}$ obviously satisfy (C5). Besides, it is not difficult to get that the following implication operators (from Ref. 22) also satisfy (C5).

$$R_{18}(a,b) = \begin{cases} b, & a \le b, \\ 0, & a > b, \end{cases} \quad R_{19}(a,b) = 1 - ab, \quad R_{20}(a,b) = (a \wedge (1-a)) \vee b.$$

Therefore, if $\to_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, and $\to_1 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}, \ldots, R_{20}\}$, then the response function is a step response function (from Corollaries 3.1(i) and 3.2(i)), which means that such SISO fuzzy system via the universal triple I method can hardly be used in practical systems.

Remark 3.6. (i) (C7) obviously holds for R_M, R_{La} and the following implication operators (from Ref. 23).

$$R_{21}(a,b) = \begin{cases} 0, & a+b \le 1, \\ a, & \text{else}, \end{cases} \quad R_{22}(a,b) = \begin{cases} 0, & a+b \le 1, \\ a \wedge b, & \text{else}, \end{cases}$$
$$R_{23}(a,b) = \begin{cases} 0, & b < 1, \\ a, & \text{else}, \end{cases} \quad R_{24}(a,b) = \begin{cases} 0, & a+b \le 1, \\ (a+b-1)/b, & \text{else}, \end{cases}$$
$$R_{25}(a,b) = \begin{cases} 0, & b = 0, \\ a^{1/b}, & \text{else}, \end{cases} \quad R_{26}(a,b) = \begin{cases} 0, & b = 0, \\ a, & \text{else}, \end{cases}$$
$$R_{27}(a,b) = ab/[1+(1-a)(1-b)], \quad R_{28}(a,b) = (a^p + b^p - 1)^{1/p} \vee 0 (p > 0).$$

If $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, and $\rightarrow_1 \in \{R_M, R_{La}, R_{21}, R_{22}, \ldots, R_{28}\}$, then $\rightarrow_1$ satisfies (C7). Thus the SISO fuzzy system via the universal triple I method is approximately an interpolation function (which may demand $x^* \in E_y$) according to Corollaries 3.1(ii) and 3.2(ii). Hence it can be universal approximator and then usable in practice.

(ii) (C6) obviously holds for $R_{29}, R_{30}, R_{31}, R_{32}$ (as follows), and (C8) evidently holds for $R_{33}, R_{34}, R_{35}, R_{36}$ (as follows), in which new implication operators are from Ref. 22.

$$R_{29}(a,b) = (a+b)/2, \quad R_{30}(a,b) = (1+a)/2 - ab,$$
$$R_{31}(a,b) = (1-a)/2 + ab, \quad R_{32}(a,b) = (1+a-b)/2,$$
$$R_{33}(a,b) = a(1-b), \quad R_{34}(a,b) = 0 \vee (a-b),$$
$$R_{35}(a,b) = (a-ab)/(1+b-ab), \quad R_{36}(a,b) = (1-a)(1-b-ab) \vee 0.$$

If $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ and $\rightarrow_1 \in \{R_{29}, R_{30}, \ldots, R_{36}, R_Z\}$, then $\rightarrow_1$ satisfies (C6) or (C8), or $\rightarrow_1 = R_Z$, thus the SISO fuzzy system via the universal triple I method is approximately a fitted function (which may demand $x^* \in E_y$) according to Corollaries 3.1(iii)(iv), 3.2(iii)(iv), and then it may be usable in practical systems.

To sum up, when $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ and $\rightarrow_1 \in \{R_M, R_{La}, R_Z, R_{21}, R_{22}, \ldots, R_{36}\}$, the corresponding fuzzy system can be practicable. Thus, 190 usable SISO fuzzy systems via the universal triple I method are obtained in this paper.

Note that the universal triple I method degenerates into the CRI method if $\rightarrow_2 = R_M$ (by Remark 2.2), then we can get the response functions of related SISO fuzzy systems via the CRI method from Corollary 3.1, Remarks 3.5 and 3.6. In detail, the SISO fuzzy system via the CRI method is approximately a step response function when $\rightarrow_1 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}, \ldots, R_{20}\}$, which can hardly be used in practical systems. Moreover, the SISO fuzzy system via the CRI method is approximately an interpolation function or a fitted function when $\rightarrow_1 \in \{R_M, R_{La}, R_Z, R_{21}, R_{22}, \ldots, R_{36}\}$, which implies that such fuzzy system can be usable in practical systems.

Remark 3.7. From Ref. 23, the SISO fuzzy system via the CRI method is approximately an interpolation function if $\rightarrow \in \{R_{21}, R_{22}, \ldots, R_{28}\}$, and a fitted function if $\rightarrow \in \{R_{29}, R_{31}\}$, thus such 10 SISO fuzzy systems via the CRI method are usable. It is easy to find that these results of Ref. 23 are the same as the related ones in this paper. Moreover, besides these implication operators, we draw the conclusions that the SISO fuzzy systems via the CRI method employing $R_M, R_{La}, R_Z, R_{30}, R_{32}, R_{33}, \ldots, R_{36}$ are also usable. Thus, in this paper, we provide more usable fuzzy systems via the CRI method (than Ref. 23).

If $\rightarrow_1 = \rightarrow_2$, the universal triple I method degenerates into the triple I method, then we can get Corollary 3.4.

Corollary 3.4. *In the SISO fuzzy system via the triple I method,*

(i) *if* $\rightarrow \in \{R_0, R_L, R_{10}, R_G, R_{Go}, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, *then the SISO fuzzy system is approximately a step response function*;
(ii) *if* $\rightarrow$ *takes* R_M, *then the SISO fuzzy system is approximately a univariate piecewise interpolation function*;
(iii) *if* $\rightarrow$ *takes* R_Z, *then the SISO fuzzy system is approximately a univariate piecewise fitted function for the case of* $x^* \in E_y$, *and a step response function for the case of* $x^* \in X - E_y$.

Remark 3.8. From Ref. 22, only two fuzzy systems are usable (where R_M or R_Z is employed) in 51 SISO fuzzy systems via the triple I method. Such conclusions are the same as the related ones in Corollary 3.4 in this paper.

4. DISO Fuzzy Systems Based on the Universal Triple I Method and Their Response Functions

In the previous researches related to the fuzzy systems via the CRI method or triple I method, it is common to discuss SISO fuzzy systems and DISO fuzzy systems. Therefore, we shall investigate the DISO fuzzy systems based on universal triple I method.

4.1. *Construction of the DISO fuzzy systems via the universal triple I method*

The DISO fuzzy systems via the universal triple I method shall be constructed.

Let X and Y be the universe of inputs x and y, respectively, and Z the universe of output z. Denote

$$\mathbb{A} = \{A_i\}_{(1 \le i \le n)}, \quad \mathbb{B} = \{B_i\}_{(1 \le i \le n)}, \quad \text{and} \quad \mathbb{C} = \{C_i\}_{(1 \le i \le n)},$$

where $A_i \in F(X)$, $B_i \in F(Y)$, $C_i \in F(Z)$ in which $F(X), F(Y), F(Z)$ are the sets of all fuzzy subsets on X, Y, Z, respectively. We regard $\mathbb{A}, \mathbb{B}, \mathbb{C}$ as linguistic variables, then the fuzzy reasoning rules can be expressed as follows:

$$\text{If } x \text{ is } A_i \text{ and } y \text{ is } B_i, \text{ then } z \text{ is } C_i, \ i = 1, \ldots, n, \tag{11}$$

where $x \in X, y \in Y, z \in Z$ are called base variables.

Similar to Refs. 8, 22 and 23 and the case of SISO fuzzy system, the reasoning relation of ith inference rule can be changed into $(A_i(x) \wedge B_i(y)) \rightarrow_1 C_i(z)$, and we get the whole reasoning rule

$$\rho_1(x, y, z) \triangleq \vee_{i=1}^{n} ((A_i(x) \wedge B_i(y)) \rightarrow_1 C_i(z)).$$

Given $A^* \in F(X), B^* \in F(Y)$, the reasoning conclusion $C^* \in F(Z)$ can be obtained by the universal triple I method of fuzzy reasoning. Therefore, (2) should be changed into:

$$\rho_1(x, y, z) \rightarrow_2 ((A^*(x) \wedge B^*(y)) \rightarrow_2 C^*(z)). \tag{12}$$

It is similar to the case of SISO fuzzy system, we can get the MinP-solutions derived from (12). We also analyze the case $\rightarrow_2 = R_0$ as an example. For (2), the MinP-solution is

$$B^*(y) = \sup_{x \in E_y} \{A^*(x) \wedge (A(x) \rightarrow_1 B(y))\},$$

where $E_y = \{x \in X | (A^*(x))' < A(x) \rightarrow_1 B(y)\}$. If we compare (2) with (12), it is easy to get that $(A(x) \rightarrow_1 B(y))$, $A^*(x)$ and $B^*(y)$ are respectively displaced by $\rho_1(x, y, z)$, $(A^*(x) \wedge B^*(y))$ and $C^*(z)$. Thus, we can similarly obtain that the MinP-solution from (12) is

$$C^*(z) = \sup_{(x,y) \in E_z} \{(A^*(x) \wedge B^*(y)) \wedge \rho_1(x, y, z)\},$$

where $E_z = \{(x, y) \in X \times Y | (A^*(x) \wedge B^*(y))' < \rho_1(x, y, z)\}$.

Now we shall construct the DISO fuzzy system via the universal triple I method as the following three steps:

(i) For a DISO fuzzy system, the input value is a crisp quantity $(x^*, y^*) \in X \times Y$. So we should treat (x^*, y^*) by fuzzier (still using singleton fuzzier), and get

$$A^*_{x^*} \triangleq \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases} \text{ and } B^*_{y^*} \triangleq \begin{cases} 1, & y = y^* \\ 0, & y \neq y^* \end{cases}.$$

(ii) Then we achieve C^* by the universal triple I method of fuzzy reasoning (from the inputs $A^*_{x^*}$ and $B^*_{y^*}$).

(iii) Similar to Sec. 3.1, we mainly adopt the centroid defuzzier, i.e.,

$$z^* = \frac{\int_Z zC^*(z)\,dz}{\int_Z C^*(z)\,dz};$$

and when $C^*(z) \equiv 0$, we utilize the defuzzier of average from the maximum (which takes $hgt(C^*) = \{z \in Z \mid C^*(z) = \sup_{z \in Z} C^*(z)\}$ and then $z^* = \int_{hgt(Z)} zdz / \int_{hgt(Z)} dz$, where $hgt(Z) = Z$ here).

To sum up, there is an output $z^* = G(x^*, y^*)$ for each input (x^*, y^*). Then a DISO fuzzy system via the universal triple I method is constructed.

To discuss the response functions of DISO fuzzy systems, suppose $\mathbb{A} = \{A_i\}_{(1\leq i\leq n)}$, $\mathbb{B} = \{B_i\}_{(1\leq i\leq n)}$ and $\mathbb{C} = \{C_i\}_{(1\leq i\leq n)}$ are respectively the fuzzy partitions on X, Y and Z (where A_i, B_i, C_i are integrable functions). We assume that X, Y and Z are all real number intervals, e.g., $X = [a, b]$, $Y = [c, d]$ and $Z = [e, f]$ in which $a < x_1 < x_2 < \cdots < x_n < b$, $c < y_1 < y_2 < \cdots < y_n < d$ and $e < z_1 < z_2 < \cdots < z_n < f$, where x_i, y_i, z_i are respectively peak-points of A_i, B_i, C_i and corresponding distributions (for $\{a, x_1, x_2, \ldots, x_n, b\}$, $\{c, y_1, y_2, \ldots, y_n, d\}$, $\{e, z_1, z_2, \ldots, z_n, f\}$,) are basically uniform.

Let $h_1 = z_1 - e$, $h_i = z_i - z_{i-1}$ $(i = 2, 3, \ldots, n)$ and $h = \max_{1\leq i\leq n}\{h_i\}$. Since $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are all fuzzy partitions, they have Kronecker property: $A_i(x_j) = B_i(y_j) = C_i(z_j) = \delta_{ij}$. By the definition of definite integral, we obtain for the centroid defuzzier:

$$z^* = \frac{\int_Z zC^*(z)\,dz}{\int_Z C^*(z)\,dz} \approx \frac{\sum_{i=1}^n z_i C^*(z_i)h_i}{\sum_{i=1}^n C^*(z_i)h_i}. \tag{13}$$

Similarly, we get for the defuzzier of average from the maximum:

$$z^* = \frac{\int_{hgt(Z)} zdz}{\int_{hgt(Z)} dz} \approx \frac{\sum_{i=1}^n z_i h_i}{\sum_{i=1}^n h_i} \triangleq d_0.$$

4.2. *Response functions of DISO fuzzy systems via the universal triple I method*

In this subsection, for the established DISO fuzzy systems via the universal triple I method, their response functions shall be analyzed.

The equivalent form of MinP-solution C^* in the DISO fuzzy system is going to be researched (see Theorem 4.1).

Theorem 4.1. (i) *Let* $\rightarrow_2 \in \{R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, *then* $C^*(z) = \rho_1(x^*, y^*, z)$ *in a DISO fuzzy system via the universal triple I method.*

(ii) *Let* $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}\}$, *then* $C^*(z) = \rho_1(x^*, y^*, z)$ *when* $(x^*, y^*) \in E_z$, *and* $C^*(z) = 0$ *when* $(x^*, y^*) \notin E_z$ *in a DISO fuzzy system via the universal triple I method.*

Proof. (i) Suppose $\rightarrow_2 \in \{R_G, R_{Go}, R_M, R_{ep}\}$. We only prove the case of R_G as an example. It is similar to Theorem 3.1(i) that the MinP-solution can be expressed as

$$C^*(z) = \sup_{(x,y)\in X\times Y}\{A^*(x) \wedge B^*(y) \wedge \rho_1(x, y, z)\},$$

where $\rho_1(x, y, z) = \vee_{i=1}^n((A_i(x) \wedge B_i(y))\rightarrow_1 C_i(z))$. As for input (x^*, y^*), we get

$$A^*_{x^*} = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases} \text{ and } B^*_{y^*} = \begin{cases} 1, & y = y^* \\ 0, & y \neq y^* \end{cases}.$$

Thus it is evident to get $C^*(z) = \rho_1(x^*, y^*, z)$.

Suppose $\rightarrow_2 \in \{R_{y-0.5}, R_{dp-\beta}\}$, we only prove the case of $R_{y-0.5}$ as an example. It is similar to get that the MinP-solution can be expressed as

$$\begin{aligned} C^*(z) &= \sup_{(x,y)\in X\times Y} \{(A^*(x) \wedge B^*(y)) \otimes_{y-0.5} \rho_1(x,y,z)\} \\ &= \sup_{(x,y)\in E_z} \{1 - (\sqrt{1-(A^*(x)\wedge B^*(y))} + \sqrt{1-\rho_1(x,y,z)})^2\}, \end{aligned}$$

where

$$E_z = \{(x,y) \in X \times Y \mid \sqrt{1-(A^*(x)\wedge B^*(y))} + \sqrt{1-\rho_1(x,y,z)} \le 1\},$$

$\rho_1(x,y,z) = \vee_{i=1}^{n}((A_i(x) \wedge B_i(y)) \rightarrow_1 C_i(z))$. As for input (x^*, y^*), we get $A^*_{x^*}$ and $B^*_{y^*}$. It is easy to get $(x^*, y^*) \in E_z$. Note that $\otimes_{y-0.5}$ is a t-norm, then $1 \otimes_{y-0.5} b = b$ and $0 \otimes_{y-0.5} b = 0$ hold for any $b \in [0,1]$. Therefore,

$$(A^*(x) \wedge B^*(y)) \otimes_{y-0.5} \rho_1(x,y,z) = 1 \otimes_{y-0.5} \rho_1(x^*,y^*,z) = \rho_1(x^*,y^*,z)$$

when $(x,y) = (x^*, y^*)$; and

$$(A^*(x) \wedge B^*(y)) \otimes_{y-0.5} \rho_1(x,y,z) = 0 \otimes_{y-0.5} \rho_1(x,y,z) = 0$$

when $(x,y) \in X \times Y - \{(x^*, y^*)\}$. Hence

$$C^*(z) = \sup_{(x,y)\in X\times Y} \{(A^*(x) \wedge B^*(y)) \otimes_{y-0.5} \rho_1(x,y,z)\} = \rho_1(x^*,y^*,z).$$

(ii) We only prove the case of $\rightarrow_2 = R_0$ as an example. It is similar to Theorem 3.1(ii) that we have the MinP-solution

$$C^*(z) = \sup_{(x,y)\in E_z} \{A^*(x) \wedge B^*(y) \wedge \rho_1(x,y,z)\}$$

where $E_z = \{(x,y) \in X \times Y \mid (A^*(x) \wedge B^*(y))' < \rho_1(x,y,z)\}$ and $\rho_1(x,y,z) = \vee_{i=1}^{n}((A_i(x) \wedge B_i(y)) \rightarrow_1 C_i(z))$. As for input (x^*, y^*), we get $A^*_{x^*}$ and $B^*_{y^*}$. If $(x^*, y^*) \in E_z$, then we obtain $E_z = \{(x^*, y^*)\}$ by the structure of E_z, and thus $C^*(z) = \rho_1(x^*, y^*, z)$. If $(x^*, y^*) \notin E_z$, then $E_z = \emptyset$ and hence $C^*(z) = 0$. □

In the light of the conditions where $\rightarrow_1$ satisfies together with the characteristics of fuzzy partition, Lemma 4.1 can be obtained. It is found that (C5)–(C8) in Lemma 4.1 are respectively significant conditions for $\rightarrow_1$ to research the response function of the DISO fuzzy system constructed by the universal triple I method.

Lemma 4.1. *In a DISO fuzzy system via the universal triple I method,*

(i) *if $\rightarrow_1$ satisfies $(C5)$ or $(C6)$, then $\rho_1(x^*, y^*, z_j) > 0$ holds for any $(x^*, y^*) \in X \times Y$ $(j = 1, \ldots, n)$;*
(ii) *if $\rightarrow_1 = R_Z$, then $\rho_1(x^*, y^*, z_j) > 0$ holds for any $(x^*, y^*) \in X \times Y$ $(j = 1, \ldots, n)$;*
(iii) *if $\rightarrow_1$ satisfies $(C7)$ or $(C8)$, then there exists z_j such that $\rho_1(x^*, y^*, z_j) > 0$ holds for any $(x^*, y^*) \in X \times Y$ $(j \in \{1, \ldots, n\})$.*

The response functions of DISO fuzzy systems via the universal triple I method are obtained in Theorems 4.2–4.4 (from the equivalent form of MinP-solution and the condition that $\rightarrow_1$ satisfies).

Theorem 4.2. *Suppose that the MinP-solution is* $C^*(z) = \rho_1(x^*, y^*, z)$, *and that* $\rightarrow_1$ *satisfies* $(C7)$ *in a DISO fuzzy system via the universal triple I method. Then there exists a group of base functions* $\Phi = \{\varphi_i\}_{(1\le i\le n)}$ *such that the DISO fuzzy system is approximately a binary piecewise interpolation function taking* φ_i *as its base functions* (*i.e.*, $G(x, y) = \sum_{i=1}^{n} \varphi_i(x, y)z_i$).

Proof. As for $C^*(z) = \rho_1(x^*, y^*, z) = \vee_{i=1}^{n}((A_i(x^*) \wedge B_i(y^*))\rightarrow_1 C_i(z))$, since $\rightarrow_1$ satisfies (C7) and $C_k(z_i) = \delta_{ki}$, it follows from (13) that we obtain:

$$
\begin{aligned}
z^* &\approx \frac{\sum_{i=1}^{n} z_i C^*(z_i)h_i}{\sum_{i=1}^{n} C^*(z_i)h_i} = \frac{\sum_{i=1}^{n} z_i[\vee_{k=1}^{n}((A_k(x^*) \wedge B_k(y^*))\rightarrow_1 C_k(z_i))]h_i}{\sum_{i=1}^{n}[\vee_{k=1}^{n}((A_k(x^*) \wedge B_k(y^*))\rightarrow_1 C_k(z_i))]h_i} \\
&= \frac{\sum_{i=1}^{n} z_i(A_i(x^*) \wedge B_i(y^*))h_i}{\sum_{i=1}^{n}(A_i(x^*) \wedge B_i(y^*))h_i},
\end{aligned}
\tag{14}
$$

where there exists z_i such that $C^*(z_i) = \rho_1(x^*, y^*, z_i) > 0$ $(i \in \{1, \dots, n\})$ by Lemma 4.1(iii), so $\sum_{i=1}^{n} C^*(z_i)h_i > 0$ and then (14) makes sense.

Denote

$$
C_i(x^*, y^*) \triangleq A_i(x^*) \wedge B_i(y^*),
$$

$$
\varphi_i(x^*, y^*) \triangleq h_i C_i(x^*, y^*) \Big/ \left[\sum_{i=1}^{n} h_i C_i(x^*, y^*)\right],
$$

then we get $z^* \approx \sum_{i=1}^{n} \varphi_i(x^*, y^*)z_i$. Let $\Phi \triangleq \{\varphi_i\}_{(1\le i\le n)}$, $G(x, y) \triangleq \sum_{i=1}^{n} \varphi_i(x, y)z_i$. Considering $A_k(x_i) = B_k(y_i) = \delta_{ki}$, we get

$$
\begin{aligned}
G(x_i, y_i) &= \left[\sum_{k=1}^{n} z_k(A_k(x_i) \wedge B_k(y_i))h_k\right] \Big/ \left[\sum_{k=1}^{n} (A_k(x_i) \wedge B_k(y_i))h_k\right] \\
&= z_i h_i / h_i = z_i
\end{aligned}
$$

for $i = 1, \dots, n$, then $G(x, y)$ is a binary piecewise interpolation function which regards φ_i as its base functions. □

Similar to Theorems 3.3 and 4.2, we can prove Theorem 4.3, which analyzes the case that $\rightarrow_1$ satisfies (C6) or (C8), or $\rightarrow_1 = R_Z$.

Theorem 4.3. *Suppose that the MinP-solution is* $C^*(z) = \rho_1(x^*, y^*, z)$, *and that* $\rightarrow_1$ *satisfies* $(C6)$ *or* $(C8)$, *or* $\rightarrow_1 = R_Z$ *in a DISO fuzzy system via the universal triple I method. Then there exists a group of base functions* $\Phi = \{\varphi_i\}_{(1\le i\le n)}$ *such that the DISO fuzzy system is approximately a binary piecewise fitted function regarding* φ_i *as its base functions* (*i.e.*, $G(x, y) = \sum_{i=1}^{n} \varphi_i(x, y)z_i$).

It is similar to Theorems 3.4 and 4.2 that Theorem 4.4 can be obtained, which researches the case corresponding to the step response function.

Theorem 4.4. (i) *Suppose that the MinP-solution is* $C^*(z) = \rho_1(x^*, y^*, z)$, *and that* $\to_1$ *satisfies* $(C5)$ *in a DISO fuzzy system via the universal triple I method. Then the DISO fuzzy system is approximately a step response function* (*i.e.*, $G(x, y) = d_0$).

(ii) *Suppose that the MinP-solution is* $C^*(z) = a(a \in [0, 1])$ *in a DISO fuzzy system via the universal triple I method, then the DISO fuzzy system is approximately a step response function* (*i.e.*,$G(x, y) = d_0$).

Following that, when $\to_2$ employs specific implication operator in DISO fuzzy systems via the universal triple I method, we can similarly achieve Corollaries 4.1 and 4.2.

Corollary 4.1. *Suppose that* $\to_2 \in \{R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ *in a DISO fuzzy system via the universal triple I method.*

- (i) *Let* $\to_1$ *satisfy* (*C5*) (*especially* $\to_1 \in \{R_G, R_L,\ R_0,\ R_{Go},\ R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}, \ldots, R_{20}\}$), *then the DISO fuzzy system is approximately a step response function.*
- (ii) *Let* $\to_1$ *satisfy* (*C7*) (*especially* $\to_1 \in \{R_M, R_{La}, R_{21}, R_{22}, \ldots, R_{28}\}$), *then the conclusion is the same as Theorem* 4.2.
- (iii) *Let* $\to_1$ *satisfy* (*C6*) *or* (*C8*), *or* $\to_1 = R_Z$ (*especially* $\to_1 \in \{R_{29}, R_{30}, \ldots, R_{36}, R_Z\}$), *then the conclusion is the same as Theorem* 4.3.

Corollary 4.2. *Suppose that* $\to_2 \in \{R_0,\ R_L, R_Z,\ R_{10}\}$ *in a DISO fuzzy system via the universal triple I method.*

- (i) *Let* $\to_1$ *satisfy* (*C5*) (*especially* $\to_1 \in \{R_G,\ R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}, \ldots, R_{20}\}$), *then the DISO fuzzy system is approximately a step response function.*
- (ii) *Let* $\to_1$ *satisfy* (*C7*) (*especially* $\to_1 \in \{R_M, R_{La}, R_{21}, R_{22}, \ldots, R_{28}\}$), *then there are two cases to be considered*: (*a*) *Suppose* $(x^*, y^*) \in E_z$, *then the conclusion is the same as Theorem* 4.2. (*b*) *Suppose* $(x^*, y^*) \notin E_z$, *then the DISO fuzzy system is approximately a step response function.*
- (iii) *Let* $\to_1$ *satisfy* (*C6*) *or* (*C8*), *or* $\to_1 = R_Z$ (*especially* $\to_1 \in \{R_{29},\ R_{30}, \ldots, R_{36}, R_Z\}$), *then there are two cases to be considered*: (*a*) *Suppose* $(x^*, y^*) \in E_z$, *then the conclusion is the same as Theorem* 4.3. (*b*) *Suppose* $(x^*, y^*) \notin E_z$, *then the DISO fuzzy system is approximately a step response function.*

Remark 4.1. It is similar to Remark 3.5 that if we take $\to_2 \in \{R_0,\ R_L,\ R_Z,\ R_{10}, R_G, R_{Go},\ R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ and $\to_1 \in \{R_G,\ R_L, sR_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep},\ R_{y-0.5}, R_{dp-\beta},\ R_{10},\ R_{16},\ R_{17}, \ldots, R_{20}\}$, then the DISO fuzzy system via the universal triple I method is approximately a step response function, which can hardly be used in practical systems. It is similar to Remark 3.6 that if $\to_2 \in \{R_0,\ R_L, R_Z, R_{10}, R_G,\ R_{Go}, R_M,\ R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ and $\to_1 \in \{R_Z, R_M, R_{La}, R_{21}, R_{22}, \ldots, R_{36}\}$, the DISO fuzzy system via the universal triple I method is approximately a fitted function or an interpolation function (which may demand $(x^*, y^*) \in E_z$), thus it can be usable in practice. As a result, 190 usable DISO fuzzy systems via the universal triple I method are provided. Moreover, it is easy to find that the DISO fuzzy system via the universal triple I method is basically consistent with the SISO fuzzy system via the universal triple I method from the viewpoint of response function.

Take into account that the universal triple I method degenerates into the CRI method if $\rightarrow_2 = R_M$, then we can get the response functions of related DISO fuzzy systems via the CRI method from Corollary 4.1 and Remark 4.1. In detail, the DISO fuzzy system via the CRI method is approximately a step response function where $\rightarrow_1 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}, \ldots, R_{20}\}$, which can hardly be used in practice. Moreover, the DISO fuzzy system via the CRI method is approximately a binary piecewise interpolation function or fitted function where $\rightarrow_1 \in \{R_M, R_{La}, R_Z, R_{21}, R_{22}, \ldots, R_{36}\}$, meaning that such DISO fuzzy system via the CRI method can be usable in practical systems.

When $\rightarrow_1 = \rightarrow_2$, the universal triple I method degenerates into the triple I method, thus we can get Corollary 4.3.

Corollary 4.3. *In the DISO fuzzy system via the triple I method*,

(i) *if* $\rightarrow \in \{R_0, R_L, R_{10}, R_G, R_{Go}, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, *then the corresponding DISO fuzzy system is approximately a step response function*;

(ii) *if* $\rightarrow$ *takes* R_M, *then the DISO fuzzy system is approximately a binary piecewise interpolation function*;

(iii) *if* $\rightarrow$ *takes* R_Z, *then the DISO fuzzy system is approximately a binary piecewise fitted function for the case of* $(x^*, y^*) \in E_z$, *and a step response function for the case of* $(x^*, y^*) \notin E_z$.

Remark 4.2. From Ref. 23, the DISO fuzzy system via the CRI method is approximately a binary piecewise interpolation function if $\rightarrow \in \{R_{21}, R_{22}, \ldots, R_{28}\}$, and a binary piecewise fitted function if $\rightarrow \in \{R_{29}, R_{31}\}$, hence such 10 DISO fuzzy systems via the CRI method are practicable, which are consistent with the case of SISO fuzzy systems via the CRI method. Similar to Remark 3.7, the related conclusions of the DISO fuzzy systems via the CRI method (in this paper) are the same as the ones of Ref. 23, and this paper provides more usable DISO fuzzy systems via the CRI method (e.g., taking $R_M, R_{La}, R_Z, R_{30}, R_{32}, R_{33}, \ldots, R_{36}$).

Remark 4.3. From Ref. 22, only two DISO fuzzy systems are usable (where R_M or R_Z is employed) in 51 DISO fuzzy systems via the triple I method. Such conclusions are consistent with the related ones in Corollary 4.3 in this paper.

Remark 4.4. In Ref. 27, only the case of SISO fuzzy system via the universal triple I method was considered (with a few preliminary conclusions). However, in this paper we discuss the cases of both SISO and DISO fuzzy systems via the universal triple I method, and point out that the conclusions (from the viewpoint of response function) of such two cases are basically consistent.

Remark 4.5. In Secs. 3 and 4, the fuzzy systems via universal triple I method are analyzed, where

$$\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\} \text{ (altogether 10 operators)}$$

and

$$\rightarrow_1 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta},$$

$$R_{10}, R_Z, R_M, R_{La}, R_{16}, R_{17}, \ldots, R_{36}\} \text{ (altogether 36 operators).}$$

Thus there are altogether $10 * 36 = 360$ fuzzy systems via universal triple I method. It can be divided into three cases (where $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$):

(a) When $\rightarrow_1 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}, \ldots, R_{20}\}$, the response function is a step response function, and such $10 * 17 = 170$ fuzzy systems can hardly be used.
(b) When $\rightarrow_1 \in \{R_M, R_{La}, R_{21}, R_{22}, \ldots, R_{28}\}$, such $10 * 10 = 100$ fuzzy systems are approximately interpolation functions, and can be usable in practice.
(c) When $\rightarrow_1 \in \{R_{29}, R_{30}, \ldots, R_{36}, R_Z\}$, such $10 * 9 = 90$ fuzzy systems are approximately fitted functions, and then may be usable in practical systems.
As a result, $100 + 90 = 190$ fuzzy systems via the universal triple I method are usable.

Remark 4.6. From the results mentioned above, we draw the conclusions that when $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, 190 fuzzy systems via the universal triple I method are usable, and 19 fuzzy systems via the CRI method are practicable, and two fuzzy systems via the triple I method are usable. Therefore, in the scope of $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, usable fuzzy systems via the universal triple I method are more than ones via the CRI method or triple I method. Thus, the universal triple I method has bigger effective choosing space.

We shall give some examples of DISO fuzzy systems based on the universal triple I method, where $\rightarrow_2 = R_L, \rightarrow_1 = R_M$ (which has excellent response ability). And we take $\mathbb{A} = \{A_i\}_{(1\le i\le 5)}, \mathbb{B} = \{B_i\}_{(1\le i\le 5)}$, and $\mathbb{C} = \{C_i\}_{(1\le i\le 5)}$ where $A_i \in F(X)$, $B_i \in F(Y)$, $C_i \in F(Z)$ are triangular fuzzy sets in which $X = Y = Z = [0, 1]$. The peak-points of $A_1, A_2, \ldots, A_5$ are $0, 0.25, 0.5, 0.75, 1$ respectively, and the expressions of $A_1, A_2, \ldots, A_5$ are as follows (and the ones of $B_1, B_2, \ldots, B_5$ together with $C_1, C_2, \ldots, C_5$ are similar):

$$A_1(x) = \begin{cases} (0.25 - x)/0.25, & 0 < x \le 0.25, \\ 0, & \text{else,} \end{cases}$$

$$A_2(x) = \begin{cases} x/0.25, & 0 < x \le 0.25, \\ (0.5 - x)/0.25, & 0.25 < x \le 0.5, \\ 0, & \text{else,} \end{cases}$$

$$A_3(x) = \begin{cases} (x-0.25)/0.25, & 0.25 < x \le 0.5, \\ (0.75-x)/0.25, & 0.5 < x \le 0.75, \\ 0, & \text{else}, \end{cases}$$

$$A_4(x) = \begin{cases} (x-0.5)/0.25, & 0.5 < x \le 0.75, \\ (1-x)/0.25, & 0.75 < x \le 1, \\ 0, & \text{else}, \end{cases}$$

$$A_5(x) = \begin{cases} (x-0.75)/0.25, & 0.75 < x \le 1, \\ 0, & \text{else}. \end{cases}$$

The part rules are as follows:

(i) $A_2, B_5 \to C_1$, (ii) $A_2, B_4 \to C_2$, (iii) $A_3, B_5 \to C_1$,
(iv) $A_3, B_4 \to C_2$, (v) $A_3, B_3 \to C_3$, (vi) $A_4, B_3 \to C_3$,
(vii) $A_4, B_4 \to C_2, \ldots$.

Then some specific examples can be obtained, which are shown in Table 1. In detail, the computing process of the first and second examples in Table 1 are illustrated in Examples 4.1 and 4.2.

Example 4.1. Suppose that the input is $(x^*, y^*) = (0.45, 0.9)$. Then it is similar to Theorem 4.1 that we have

$$\begin{aligned} C^*(z) &= \rho_1(x^*, y^*, z) = \vee_{i=1}^{n}[(\overline{A}_i(x^*) \wedge \overline{B}_i(y^*)) \to_1 \overline{C}_i(z)] \\ &= \vee_{i=1}^{n}[(\overline{A}_i(0.45) \wedge \overline{B}_i(0.9)) \wedge \overline{C}_i(z)] \\ &= [(A_2(0.45) \wedge B_5(0.9)) \wedge C_1(z)] \vee [(A_2(0.45) \wedge B_4(0.9)) \wedge C_2(z)] \\ &\quad \vee[(A_3(0.45) \wedge B_5(0.9)) \wedge C_1(z)] \vee [(A_3(0.45) \wedge B_4(0.9)) \wedge C_2(z)] \\ &= [0.2 \wedge 0.6 \wedge C_1(z)] \vee [0.2 \wedge 0.4 \wedge C_2(z)] \\ &\quad \vee[0.8 \wedge 0.6 \wedge C_1(z)] \vee [0.8 \wedge 0.4 \wedge C_2(z)] \\ &= [0.6 \wedge C_1(z)] \vee [0.4 \wedge C_2(z)], \end{aligned}$$

where it relates to the rules (i)–(iv), and we get

$$C^*(z) = \begin{cases} 0.6, & 0 < z \le 0.1, \\ (0.25-z)/0.25, & 0.1 < z \le 0.15, \\ 0.4, & 0.15 < z \le 0.4, \\ (0.5-z)/0.25, & 0.4 < z \le 0.5, \\ 0, & \text{else}. \end{cases}$$

Finally, we obtain by the centroid defuzzier that

$$z^* = \frac{\int_Z zC^*(z)\,dz}{\int_Z C^*(z)\,dz} = \frac{0.04225}{0.205} = 0.2061.$$

Table 1. Some examples of DISO fuzzy systems based on the universal triple I method.

Input (x^*, y^*)	$C^*(z)$	Output z^*
$(0.45, 0.9)$	$C^*(z) = \begin{cases} 0.6, & 0 < z \leq 0.1, \\ (0.25 - z)/0.25, & 0.1 < z \leq 0.15, \\ 0.4, & 0.15 < z \leq 0.4, \\ (0.5 - z)/0.25, & 0.4 < z \leq 0.5, \\ 0, & \text{else.} \end{cases}$	0.2061
$(0.6, 0.7)$	$C^*(z) = \begin{cases} z/0.25, & 0 < z \leq 0.15, \\ 0.6, & 0.15 < z \leq 0.35, \\ (0.5 - z)/0.25, & 0.35 < z \leq 0.45, \\ 0.2, & 0.45 < z \leq 0.7, \\ (0.75 - z)/0.25, & 0.7 < z \leq 0.75, \\ 0, & \text{else.} \end{cases}$	0.3173
$(0.875, 0.1)$	$C^*(z) = \begin{cases} (z - 0.5)/0.25, & 0.5 < z \leq 0.6, \\ 0.4, & 0.6 < z \leq 0.85, \\ (z - 0.75)/0.25, & 0.85 < z \leq 0.875, \\ 0.5, & 0.875 < z \leq 1, \\ 0, & \text{else.} \end{cases}$	0.7852
$(0.45, 0.25)$	$C^*(z) = \begin{cases} (z - 0.25)/0.25, & 0.25 < z \leq 0.3, \\ 0.2, & 0.3 < z \leq 0.55, \\ (z - 0.5)/0.25, & 0.55 < z \leq 0.7, \\ 0.8, & 0.7 < z \leq 0.8, \\ (1 - z)/0.25, & 0.8 < z \leq 1, \\ 0, & \text{else.} \end{cases}$	0.6897
$(0.35, 0.85)$	$C^*(z) = \begin{cases} 0.4, & 0 < z \leq 0.1, \\ z/0.25, & 0.1 < z \leq 0.15, \\ 0.6, & 0.15 < z \leq 0.35, \\ (0.5 - z)/0.25, & 0.35 < z \leq 0.5, \\ 0, & \text{else.} \end{cases}$	0.2312
$(1.0, 0.2)$	$C^*(z) = \begin{cases} (z - 0.5)/0.25, & 0.5 < z \leq 0.7, \\ 0.8, & 0.7 < z \leq 0.8, \\ (1 - z)/0.25, & 0.8 < z \leq 0.95, \\ 0.2, & 0.95 < z \leq 1, \\ 0, & \text{else.} \end{cases}$	0.7548
$(0.625, 0.6)$	$C^*(z) = \begin{cases} z/0.25, & 0 < z \leq 0.1, \\ 0.4, & 0.1 < z \leq 0.35, \\ (z - 0.25)/0.25, & 0.35 < z \leq 0.375, \\ 0.5, & 0.375 < z \leq 0.625, \\ (0.75 - z)/0.25, & 0.625 < z \leq 0.75, \\ 0, & \text{else.} \end{cases}$	0.3870
$(0.65, 0.4)$	$C^*(z) = \begin{cases} (z - 0.25)/0.25, & 0.25 < z \leq 0.4, \\ 0.6, & 0.4 < z \leq 0.6, \\ (0.75 - z)/0.25, & 0.6 < z \leq 0.65, \\ 0.4, & 0.65 < z \leq 0.9, \\ (1 - z)/0.25, & 0.9 < z \leq 1, \\ 0, & \text{else.} \end{cases}$	0.6048

Example 4.2. Suppose that the input is $(x^*, y^*) = (0.6, 0.7)$. Then we can get

$$\begin{aligned}
C^*(z) &= \rho_1(x^*, y^*, z) = \vee_{i=1}^{n}[(\overline{A}_i(x^*) \wedge \overline{B}_i(y^*)) \to_1 \overline{C}_i(z)] \\
&= \vee_{i=1}^{n}[(\overline{A}_i(0.6) \wedge \overline{B}_i(0.7)) \wedge \overline{C}_i(z)] \\
&= [(A_3(0.6) \wedge B_4(0.7)) \wedge C_2(z)] \vee [(A_3(0.6) \wedge B_3(0.7)) \wedge C_3(z)] \\
&\quad \vee [(A_4(0.6) \wedge B_4(0.7)) \wedge C_2(z)] \vee [(A_4(0.6) \wedge B_3(0.7)) \wedge C_3(z)] \\
&= [0.6 \wedge 0.8 \wedge C_2(z)] \vee [0.6 \wedge 0.2 \wedge C_3(z)] \\
&\quad \vee [0.4 \wedge 0.8 \wedge C_2(z)] \vee [0.4 \wedge 0.2 \wedge C_3(z)] \\
&= [0.6 \wedge C_2(z)] \vee [0.2 \wedge C_3(z)],
\end{aligned}$$

where it relates to the rules (iv)–(vii), and we have

$$C^*(z) = \begin{cases} z/0.25, & 0 < z \leq 0.15, \\ 0.6, & 0.15 < z \leq 0.35, \\ (0.5 - z)/0.25, & 0.35 < z \leq 0.45, \\ 0.2, & 0.45 < z \leq 0.7, \\ (0.75 - z)/0.25, & 0.7 < z \leq 0.75, \\ 0, & \text{else}. \end{cases}$$

Lastly, we achieve by the centroid defuzzier that

$$z^* = \frac{\int_Z zC^*(z)\,dz}{\int_Z C^*(z)\,dz} = \frac{0.0825}{0.26} = 0.3173.$$

Before the end of this section, we shall analyze the duty of first implication $\to_1$ and second implication $\to_2$ in the universal triple I method. It is not difficult to know that the form of the solution of universal triple I method is basically determined only if $\to_2$ is chosen (i.e., $\to_2$ takes an implication operator), and hence $\to_2$ determines the reasoning mechanism to a large extent (see e.g., Theorem 2.2 and Proposition 2.2). Meanwhile, $\to_1$ often exists as the form of $(A(x) \to_1 B(y))$ (or $\rho_1(x, y)$ and so on), which embodies the function of rule base. What is more, the second implication has leading status for the universal triple I method in virtue of its effect on direction of inference.

Summarizing above, the second implication and first implication, respectively, embody the reasoning mechanism and function of rule base. Thus, the way which lets $\to_1, \to_2$ take different implication operators, corresponds to separating of the rule base and reasoning mechanism, which further demonstrates the reasonability of the universal triple I method.

5. Conclusions

To solve the problem that the effect of the triple I method is imperfect from the viewpoint of fuzzy systems, we generalize the triple I method to the universal triple I method, and investigate the fuzzy systems (via the universal triple I method) and their response functions. The main contributions and conclusions are as follows.

The fuzzy systems via the universal triple I method are constructed, and then the response functions of 360 fuzzy systems are discussed. We get the following results.

(a) If we take $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, and $\rightarrow_1 \in \{R_M, R_{La}, R_{21}, R_{22}, \ldots, R_{28}\}$, then the fuzzy system based on the universal triple I method is approximately an interpolation function. Hence such 100 fuzzy systems can be universal approximators and then usable in practical systems.

(b) If we employ $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$ and $\rightarrow_1 \in \{R_{29}, R_{30}, \ldots, R_{36}, R_Z\}$, then the fuzzy system based on the universal triple I method is approximately a fitted function. Thus such 90 fuzzy systems may be usable in practical systems.

(c) If we take $\rightarrow_2 \in \{R_0, R_L, R_Z, R_{10}, R_G, R_{Go}, R_M, R_{ep}, R_{y-0.5}, R_{dp-\beta}\}$, and $\rightarrow_1 \in \{R_G, R_L, R_0, R_{Go}, R_{GR}, R_{KD}, R_R, R_Y, R_{ep}, R_{y-0.5}, R_{dp-\beta}, R_{10}, R_{16}, R_{17}, \ldots, R_{20}\}$, then the response function is a step response function. Therefore such 170 fuzzy systems based on the universal triple I method can hardly be used in practice.

The results show that 190 fuzzy systems via the universal triple I method are usable, and that 19 fuzzy systems via the CRI method are practicable, and that two fuzzy systems via the triple I method are usable. Therefore, usable fuzzy systems based on the universal triple I method are more than ones based on the CRI method or triple I method. Thus, the universal triple I method has larger effective choosing space. As a result, from the viewpoint of fuzzy systems, the universal triple I method is superior to the triple I method and CRI method.

Some related conclusions of Refs. 22, 23 and 27 are improved from the viewpoints of research idea, quantity of usable fuzzy systems and so on (see Remarks 3.2, 3.4, 3.7, 3.8 and 4.2–4.4).

It is pointed out that, in the universal triple I method, the first implication and second implication respectively embody the function of rule base and the reasoning mechanism. Therefore, in the universal triple I method, there exists the idea of separating of the rule base and reasoning mechanism, further demonstrating the reasonability of the universal triple I method. Meanwhile, the universal triple I method has close relationship with the triple I method and CRI method. Thus, it is easy to find that the research of universal triple I method will help analyze the essence of the triple I method and CRI method, and further improve the development of fuzzy reasoning theory.

For the case of m-input and single-output (where m is any natural number and $m > 2$), how can we get the response functions of the corresponding fuzzy systems (via the universal triple I method)? And what happens for the case of m_1-input and m_2-output (where m_1, m_2 are any natural numbers and $m_1, m_2 > 1$)? Moreover, if we employ other fuzzier and defuzzier,[34,35] how can we design and analyze the corresponding fuzzy systems (via the universal triple I method, or the triple I method and so forth)? These problems are more complicated and will be investigated in another paper.

It should be emphasized that our research group has already carried out a lot of works revolving around affective computing and natural language processing.[1,36–41] As one of the main problems of affective computing, recognizing human emotion can be carried through by the theory of fuzzy system. Here we only provide the main idea. We can analyze and extract the main factors determining human emotion (from the input information which may be natural language, phonetic information, expressive information, brain wave and so on). Such factors can be expressed by fuzzy language and then be regarded as the input of fuzzy system; while the output is the fuzzy set which represents corresponding emotion state derived from these factors. Thus it is not difficult for us to obtain the fuzzy reasoning rules expressed as follows:

$$\text{If } x_1 \text{ is } A_1^i, x_2 \text{ is } A_2^i, \ldots, \text{ and } x_m \text{ is } A_m^i, \text{ then } y \text{ is } B_i (i = 1, \ldots, n), \tag{15}$$

where $A_1^i, A_2^i, \ldots, A_m^i$ respectively express the factors (determining human emotion), and B_i represents the corresponding emotion state. Therefore, a fuzzy system (of m-input and single-output) for recognizing human emotion can be established. This will be an important way to investigate human emotion recognition, since the fuzzy system (including fuzzy reasoning) provides an excellent way to deal with the fuzzy, uncertain characteristics which are the essence of human emotion. Such works will be our research emphases in the further research.

Acknowledgments

This work is supported by the National High-Tech Research and Development Program of China (863 Program, 2012AA011103), the National Natural Science Foundation of China (No. 61203077, No. 61070124, No. 61105076, No. 60890075), the China Postdoctoral Science Foundation (No. 2012M521218, No. 2014T70585), the Open Project of State Key Laboratory of Virtual Reality Technology and Systems of China (No. BUAA-VR-10KF-5), and the Fundamental Research Funds for the Central Universities (No.2012HGQC0011, No. 2012HGCX0001, No. 2011HGZY0004, No. 2012HGBZ0639). The authors would like to sincerely thank the editors and reviewers for their valuable comments and suggestions which have led to a substantially improved paper.

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