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On the $\alpha(u, v)$-symmetric implicational method for R- and (S, N)-implications ☆

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ABSTRACT

The sustaining degree is generalized to the two-dimensional sustaining degree, and based on it a new symmetric implicational method is proposed and investigated. To begin with, some properties of such two kinds of sustaining degrees are carefully discussed. Furthermore, the symmetric implicational principles are improved. Aiming at the FMP (fuzzy modus ponens) and FMT (fuzzy modus tollens) problems, unified forms of the new method are obtained for R-implications and (S, N)-implications. Following that, optimal solutions of the new method are obtained for eleven R- and (S, N)-implications, and four specific examples are shown which include two continuous ones and two discrete ones. Finally, it is pointed out that the new method contains related symmetric implicational methods and full implication methods as its particular cases.

1. Introduction

Fuzzy inference has been extensively applied in the areas of fuzzy control, decision-making, pattern recognition, time series analysis and other areas (see [1–3]). The fundamental model of fuzzy inference is expressed as follows:

$$\text{If } A \text{ implies } B, \quad \text{then } A^* \text{ implies } B^*. \tag{1}$$

It includes two key problems, i.e. FMP (fuzzy modus ponens) and FMT (fuzzy modus tollens):

$$\text{FMP: from } A \longrightarrow B \text{ and } A^*, \quad \text{obtain } B^*, \tag{2}$$

$$\text{FMT: from } A \longrightarrow B \text{ and } B^*, \quad \text{obtain } A^*, \tag{3}$$

in which $A, A^* \in F(U)$ and $B, B^* \in F(V)$ (where $F(U)$, $F(V)$ respectively represent the sets of all fuzzy subsets of U and V).

The CRI (compositional rule of inference) method proposed by Zadeh has been the most commonly considered strategy to construct the solutions to the FMP and FMT problems (see [4–6]), which uses a fuzzy implication to express the underlying relationship. Then, generalizing the fuzzy implication to three fuzzy implications, Wang [7] presented a so-called

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full implication method (triple I method), whose optimal solution was the smallest $B^* \in F(V)$ (or the largest $A^* \in F(U)$) making

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \tag{4}$$

employ its maximum, in which $\to$ denotes a fuzzy implication. Later, it was extended to the α-full implication method, which aimed at ($\alpha \in [0, 1]$):

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \geq \alpha. \tag{5}$$

There has been an intensive research related to the full implication method. Song et al. established the full implication restriction method and the reverse full implication method in [8,9]. Using the residual implication, some unified forms of optimal solutions of the full implication method were constructed (see [10–12]). Zhang et al. [13] introduced the concept of generalized roots of theories, and based on it researched the full implication method in four kinds of propositional logic systems. Pei [14] provided a sound logical foundation for the full implication method with the monoidal t-norm based logical system. Zheng et al. [15] presented the unified form of residual intuitionistic fuzzy implications, and then investigated the full implication method for intuitionistic fuzzy sets. Later, the robustness becomes an intensive research area to study in case of the full implication method. Dai et al. discussed the robustness of the full implication method and fully implicational restriction method [16]. Wang and Duan proposed a finer measure for appraising robustness of fuzzy inference, and investigated the robustness of logic connectives and full implication method related to the finer measurements [17]. Luo and Zhou [18] put forward the $[\alpha, \beta]$-full implication method based on interval-valued fuzzy set and demonstrated its robustness. Luo and Liu [19] presented the sensitivity interval-valued fuzzy connectives, and then investigated the robustness of interval-valued full implication method. To sum up, it has been demonstrated that the full implication method exhibits a number of sound properties (e.g., strict logic basis, reversibility properties, continuity, robustness and others).

From a detailed point of view, the first and the third fuzzy implications in (4) can be seen as the implication connective in a logic system; and the second fuzzy implication in (4) is concerned with the "if–then" relation of fuzzy inference model (1). Based on this idea, in [20], we generalized (4) as follows:

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)), \tag{6}$$

where $\to_1, \to_2$ are two fuzzy implications. The corresponding fuzzy inference method was called the basic symmetric implicational method. Meanwhile, a more general α-symmetric implicational method [20] was obtained from $\alpha \in [0, 1]$ in the following form

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \geq \alpha, \tag{7}$$

which reflected the idea of sustaining degrees. It demanded that the supporting ability of $A(u) \to_1 B(v)$ to $A^*(u) \to_1 B^*(v)$ was not less than α, which was considered with regard to the fuzzy implication $\to_2$. It was verified in [20] that the symmetric implicational method formed a reasonable fuzzy inference algorithm.

However, the α-symmetric implicational method cannot contain the basic symmetric implicational method as its special case (e.g., when the maximum of (6) is not a constant value). Moreover, the case of (6) corresponds to

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \geq W(u, v),$$

where $W(u, v)$ is the maximum of (6) at (u, v), which is approximate but different from (7). Consequently, (6) and (7) are not proper to exactly reveal the essence of the symmetric implicational idea.

Aiming at this problem, for controlling each step in the reasoning process in a flexible way, we replace α in (7) with $\alpha(u, v)$ (in which $\alpha(u, v)$ is a mapping from $U \times V$ to $[0, 1]$), thus (7) is generalized to the following form

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \geq \alpha(u, v). \tag{8}$$

This means that the two-dimensional sustaining degree (see Definition 3.2) of Q to Q^* at any point (u, v) should be greater than or equal to a corresponding value $\alpha(u, v)$ (in which $Q(u, v) \triangleq A(u) \to_1 B(v)$ and $Q^*(u, v) \triangleq A^*(u) \to_1 B^*(v)$). The previous reasoning principles are improved from the viewpoint of the two-dimensional sustaining degrees. The fuzzy inference algorithm derived from (8) is called the symmetric implicational method with two-dimensional sustaining degree, or called the $\alpha(u, v)$-symmetric implicational method.

Nowadays, the commonly used and well-studied classes of fuzzy implications are the R-implications and (S,N)-implications (see [21–23]), which are convenient to construct a unified solving framework for fuzzy inference. The aim of this paper is to research the new symmetric implicational method for these kinds of fuzzy implications.

This paper is organized as follows. Section 2 covers some preliminaries. In Section 3, the sustaining degree is generalized to the two-dimensional sustaining degree, and some properties of such two kinds of sustaining degrees are carefully analyzed. Sections 4 investigates the α(u,v)-symmetric implicational method for FMP with the emphasis on R-implications and (S,N)-implications. We improve the symmetric implicational principles, and obtain its solutions in a unified form as well as some specific cases are investigated. Section 5 covers the α(u,v)-symmetric implicational method for FMT. Section 6 shows four specific examples including two continuous ones and two discrete ones. Section 7 provides some related discussions. Section 8 offers some conclusions.

2. Preliminaries

Here we briefly recall the main concepts used throughout the study.

Definition 2.1. ([5]) A fuzzy negation is a decreasing function $N: [0,1] \to [0,1]$ which satisfies $N(0) = 1$, $N(1) = 0$. Moreover, a fuzzy negation N is said to be
(i) strict if N is continuous and strictly decreasing;
(ii) strong if N is an involution (i.e., $N(N(x)) = x$ for any $x \in [0,1]$).

Example 2.1. The classical negation $N_C(x) = 1 - x$ is a strong negation (and also a strict negation), whereas $N_K(x) = 1 - x^2$ is only a strict negation. The fuzzy negations N_{D1}, N_{D2} defined as the following

$$N_{D1}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases} \qquad N_{D2}(x) = \begin{cases} 1 & \text{if } x < 1, \\ 0, & \text{if } x = 1, \end{cases}$$

are the least and greatest ones, which are both non-strong negations. For more examples of fuzzy negations, the reader is referred to [24] or [25].

Definition 2.2. ([5]) (i) A function $T: [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm, for short), if it is increasing, commutative, associative, and has the neutral element $e = 1$.

(ii) A function $S: [0,1]^2 \to [0,1]$ is called a triangular conorm (t-conorm, for short), if it is increasing, commutative, associative, and has the neutral element $e = 0$.

Definition 2.3. ([5]) For a t-norm T and a strong negation N:

(i) the function $T_N: [0,1]^2 \to [0,1]$ is called the dual of a t-norm T w.r.t. N, where $T_N(x, y) = N(T(N(x), N(y)))$ $(x, y \in [0,1])$.

(ii) the function $S_N: [0,1]^2 \to [0,1]$ is called the dual of a t-conorm S w.r.t. N, where $S_N(x, y) = N(S(N(x), N(y)))$ $(x, y \in [0,1])$.

It is noted that T_N is a t-conorm while S_N is a t-norm.

Definition 2.4. ([26]) Let T and I be two $[0,1]^2 \to [0,1]$ mappings, (T, I) is called a residual pair or, T and I are residual to each other, if the following residuation condition holds for any $x, y, z \in [0,1]$,

$$T(x, y) \le z \quad \text{if and only if} \quad y \le I(x, z). \tag{9}$$

For a mapping I with a residual pair, the mapping T residual to I is unique, and vice versa.

Definition 2.5. ([2,11]) A fuzzy implication on $[0,1]$ is a function $I: [0,1]^2 \to [0,1]$ satisfying the following condition:
(P1) $I(0,0) = I(0,1) = I(1,1) = 1$, $I(1,0) = 0$.
We use $\mathcal{FI}$ to represent the set of all fuzzy implications. $I(a,b)$ can also be written as $a \to b$ $(a, b \in [0,1])$.

Definition 2.6. ([2]) A function $I: [0,1]^2 \to [0,1]$ is called an R-implication, if there exists a left-continuous t-norm T such that

$$I(x, y) = \sup\{t \in [0,1] \mid T(x,t) \le y\}, \quad x, y \in [0,1]. \tag{10}$$

Moreover, if an R-implication is obtained from T, then we denote it by I_T.

Proposition 2.1. *([27]) If T is a t-norm, then the following are equivalent:*
(i) T is left-continuous.
(ii) T and I_T form a residual pair, where I_T is achieved from (10).
(iii) The supremum in (10) is the maximum, that is:

$$I_T(x, y) = \max\{t \in [0,1] \mid T(x,t) \le y\}, \ x, y \in [0,1],$$

in which the right hand side always exists.

Remark 2.1. It follows from Proposition 2.1 that a t-norm T satisfies the residuation condition (9) if and only if T is left-continuous, so many authors consider R-implications only for left-continuous t-norms. It is noted that some authors also consider them for all t-norms (e.g., [24,28]).

Proposition 2.2. *([24]) Let I be an R-implication based on a left-continuous t-norm T, then I satisfies (P1) and*
(P2) $I(x,z) \geq I(y,z)$ *if* $x \leq y$,
(P3) $I(x,y) \geq I(x,z)$ *if* $y \geq z$,
(P4) $I(0,y) = 1$,
(P5) $I(x,1) = 1$,
(P6) $I(1,x) = x$,
(P7) $I(x, I(y,z)) = I(y, I(x,z))$,
(P8) $I(x,y) = 1 \Longleftrightarrow x \leq y$,
(P9) I is left-continuous w.r.t. the first variable,
(P10) I is right-continuous w.r.t. the second variable,
(P11) $I(x, T(x,y)) \geq y$,
where $x, y, z \in [0,1]$.

Proposition 2.3. *([10]) Let I be an R-implication derived from a left-continuous t-norm T, then* (T, I) *is a residual pair, and I satisfies:*
(P12) $x \leq I(y,z) \Longleftrightarrow y \leq I(x,z)$,
(P13) $I(T(x,y), z) = I(x, I(y,z))$,
(P14) $I(\sup_{x\in X} x, y) = \inf_{x\in X} I(x,y)$,
(P15) $I(x, \inf_{y\in Y} y) = \inf_{y\in Y} I(x,y)$,
where $x, y, z \in [0,1]$ *and* $X, Y \subset [0,1],\ X, Y \neq \varnothing$.

Proposition 2.4. *([24]) For a function* $I : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*
(i) I is an R-implication generated from a left-continuous t-norm.
(ii) I satisfies (P3), (P7), (P8) and (P10).

Definition 2.7. ([21,24]) A function $I : [0,1]^2 \to [0,1]$ is called an (S, N)-implication if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x), y),\ \ x, y \in [0,1]. \tag{11}$$

If N is a strong negation, then I is said to be a strong implication (S-implication). Furthermore, if an (S, N)-implication is generated from S and N, then we denote it by $I_{S,N}$.

Definition 2.8. ([22]) Let I be a fuzzy implication, then the function $N_I : [0,1] \to [0,1]$ expressed as

$$N_I(x) = I(x,0),\ \ x \in [0,1],$$

is said to be the natural negation of I.

Proposition 2.5. *([22,24]) Let I be an (S, N)-implication generated from a t-conorm S and a fuzzy negation N, then* $I \in \mathcal{FI}$ *and I satisfies (P2), (P3), (P4), (P5), (P6), (P7) and*
(P16) $N = N_I$.
Moreover, an (S,N)-implication I satisfies
(P17) $I(x,y) = I(N(y), N(x)),\ \ x, y \in [0,1]$ *(the law of contraposition w.r.t. N),*
if and only if $N = N_I$ *is a strong negation, that is, I is an S-implication.*

Proposition 2.6. *([24]) For a function* $I : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*
(i) I is an S-implication generated from a t-conorm S and a strong negation N.
(ii) I satisfies (P2) (or (P3)), (P6), (P7) and (P17) w.r.t. a strong negation N.

Proposition 2.7. *([21]) For a function* $I : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*
(i) I is an (S, N)-implication derived from some t-conorm S and some continuous (strict, strong) fuzzy negation N.
(ii) I satisfies (P2) (or (P3)), (P7) and the function N_I *is a continuous (strict, strong) fuzzy negation.*
Moreover, the representation of the (S, N)-implication is unique in this case.

Proposition 2.8. *([21]) For a function* $I : [0,1]^2 \to [0,1]$ *the following statements are equivalent:*
(i) I is a continuous (S, N)-implication.
(ii) I is an (S,N)-implication with continuous S and N.

It is easy to get Proposition 2.9 from [20].

Proposition 2.9. *If I is a fuzzy implication satisfying (P3), (P5) and (P10), then the mapping $T : [0,1]^2 \to [0,1]$ defined by*

$$T(a,b) = \inf\{x \in [0,1] \mid b \le I(a,x)\}, \quad a,b \in [0,1]$$

is residual to I.

Definition 2.9. ([29]) Let Z be any non-empty set, define partial order relation $\le_F$ on $F(Z)$ as: $A \le_F B$ if and only if $A(z_0) \le B(z_0)$ for any $z_0 \in Z$, in which $A, B \in F(Z)$.

Lemma 2.1. *([29]) $< F(Z), \le_F >$ is a complete lattice.*

3. Two-dimensional sustaining degree

Definition 3.1. ([29]) Let $\to$ be a fuzzy implication, and fuzzy sets $C, D \in F(X)$, then $C(x) \to D(x)$ is called the sustaining degree of C to D at point $x \in X$, denoted as $sust_{\to}(C,D)(x)$.

Noting that a fuzzy relation from X to Y can be regarded as a fuzzy set on $X \times Y$, it is natural to generalize the concept of sustaining degree to the two-dimensional case (see the following Definition 3.2).

Definition 3.2. Suppose that $\to$ is a fuzzy implication, and that X, Y are non-empty sets, and finally that P_1, P_2 are two fuzzy relations from X to Y. Then $P_1(x,y) \to P_2(x,y)$ is called the two-dimensional sustaining degree of P_1 to P_2 at point $(x,y) \in X \times Y$, denoted as $sust_{\to}(P_1,P_2)(x,y)$.

For the fuzzy inference mode (1), it is actually an "if–then" relationship. In detail, $A \to B$ corresponds to the "if" part, and $A^* \to B^*$ reflects the "then" part (where a fuzzy implication $\to$ is used to express "imply"). To characterize the "if–then" relationship, it is natural to hope that $A \to B$ adequately sustains $A^* \to B^*$. Here the concept of two-dimensional sustaining degree provides a measurement of the supporting ability of $A \to B$ to $A^* \to B^*$ (or more generally, the supporting ability of fuzzy relation P_1 to P_2).

Proposition 3.1. *Assume that $\to$ is a fuzzy implication satisfying (P3), (P5) and (P10), and that its residual mapping T is non-decreasing in its first component and associative. If $sust_{\to}(A,B)(x) \ge \alpha(x)$ and $sust_{\to}(B,C)(x) \ge \beta(x)$ hold for any $A, B, C \in F(X)$ and $x \in X$, then $sust_{\to}(A,C)(x) \ge T(\alpha(x), \beta(x))$, in which $\alpha(x), \beta(x) \in [0,1]$.*

Proof. Since $\to$ is an implication satisfying (P3), (P5) and (P10), from Proposition 2.9 we have that the residuation condition (9) holds. From the fact that $sust_{\to}(A,B)(x) \ge \alpha(x)$ and $sust_{\to}(B,C)(x) \ge \beta(x)$, i.e., $\alpha(x) \le A(x) \to B(x)$, $\beta(x) \le B(x) \to C(x)$, it follows from (9) that $T(A(x), \alpha(x)) \le B(x)$ and $T(B(x), \beta(x)) \le C(x)$.

By virtue of the conditions that T satisfies, it follows that

$$C(x) \ge T(B(x), \beta(x)) \ge T(T(A(x), \alpha(x)), \beta(x)) = T(A(x), T(\alpha(x), \beta(x))).$$

Thus we obtained from (9) that $T(\alpha(x), \beta(x)) \le A(x) \to C(x) = sust_{\to}(A,C)(x)$. □

Lemma 3.1. *If $\to$ is a fuzzy implication satisfying (P2), (P3), (P5) and (P10), and T the mapping residual to I, then T is non-decreasing in its first component.*

Proof. Since $\to$ satisfies (P3), (P5) and (P10), we get that the residuation condition (9) holds. Let $a_1, a_2, b, c \in [0,1]$, $a_1 \le a_2$ and $c = T(a_2, b)$. Then $T(a_2, b) \le c$ and thus $b \le a_2 \to c$ (by virtue of the residuation condition (9)). Since $\to$ satisfies (P2), we can get $b \le a_2 \to c \le a_1 \to c$, which means that $T(a_1, b) \le c = T(a_2, b)$, i.e., T is non-decreasing in its first component. □

It is easy to get Corollary 3.1 and Corollary 3.2 from Proposition 2.2, Lemma 3.1, and Proposition 3.1.

Corollary 3.1. *Assume that $\to$ is a fuzzy implication satisfying (P2), (P3), (P5) and (P10), and that its residual mapping T is associative. If $sust_{\to}(A,B)(x) \ge \alpha(x)$ and $sust_{\to}(B,C)(x) \ge \beta(x)$ hold for any $A, B, C \in F(X)$ and $x \in X$, then $sust_{\to}(A,C)(x) \ge T(\alpha(x), \beta(x))$, in which $\alpha(x), \beta(x) \in [0,1]$.*

Corollary 3.2. *Assume that $\to$ is an R-implication, and that T its residual mapping. If $sust_{\to}(A,B)(x) \ge \alpha(x)$ and $sust_{\to}(B,C)(x) \ge \beta(x)$ hold for any $A, B, C \in F(X)$ and $x \in X$, then $sust_{\to}(A,C)(x) \ge T(\alpha(x), \beta(x))$, in which $\alpha(x), \beta(x) \in [0,1]$.*

Proposition 3.2. *Assume that $\to$ is a fuzzy implication satisfying (P3), (P5) and (P10), and that its residual mapping T is non-decreasing in its first component and associative. If P_1, P_2, P_3 are three fuzzy relations from X to Y (in which X, Y are non-empty sets), and $sust_\to(P_1, P_2)(x, y) \geq \alpha(x, y)$, $sust_\to(P_2, P_3)(x, y) \geq \beta(x, y)$, then $sust_\to(P_1, P_3)(x, y) \geq T(\alpha(x, y), \beta(x, y))$, where $\alpha(x, y), \beta(x, y) \in [0, 1]$.*

Remark 3.1. It is similar to Proposition 3.1 that we can obtain the corresponding corollaries from Proposition 3.2.

Proposition 3.3. *Suppose that $\to$ is a fuzzy implication satisfying (P14) and (P15), then the following properties hold for any $A, B, A_i, B_i \in F(X)$ $(X, I \neq \varnothing,\ i \in I,\ x \in X)$:*
(i) $sust_\to(\sup_{i\in I} A_i, B)(x) = \inf_{i\in I} sust_\to(A_i, B)(x)$,
(ii) $sust_\to(A, \inf_{i\in I} B_i)(x) = \inf_{i\in I} sust_\to(A, B_i)(x)$.

Proof. Since $\to$ satisfies (P14) and (P15), it follows that

$$sust_\to(\sup_{i\in I} A_i, B)(x) = (\sup_{i\in I} A_i(x)) \to B(x) = \inf_{i\in I}(A_i(x) \to B(x)) = \inf_{i\in I} sust_\to(A_i, B)(x),$$

and

$$sust_\to(A, \inf_{i\in I} B_i)(x) = A(x) \to \inf_{i\in I} B_i(x) = \inf_{i\in I}(A(x) \to B_i(x)) = \inf_{i\in I} sust_\to(A, B_i)(x). \quad \square$$

Proposition 3.4. *Suppose that $\to$ is a fuzzy implication satisfying (P7), then*

$$sust_\to(A,\ B \to C)(x) = sust_\to(B,\ A \to C)(x)$$

holds for any $A, B, C \in F(X)$ $(X \neq \varnothing,\ x \in X)$.

Proof. Since $\to$ satisfies (P7), it follows that

$$sust_\to(A,\ B \to C)(x) = A(x) \to (B(x) \to C(x)) = B(x) \to (A(x) \to C(x)) = sust_\to(B,\ A \to C)(x). \quad \square$$

Proposition 3.5. *Suppose that $\to_1, \to_2$ are two fuzzy implications satisfying (P14) and (P15), then the following properties hold for any $A, B, C, B_i, C_i \in F(X)$ $(X, I \neq \varnothing,\ i \in I,\ x \in X)$:*
(i) $sust_{\to_1}(A,\ \sup_{i\in I} B_i \to_2 C)(x) = \inf_{i\in I} sust_{\to_1}(A,\ B_i \to_2 C)(x)$,
(ii) $sust_{\to_1}(A,\ B \to_2 \inf_{i\in I} C_i)(x) = \inf_{i\in I} sust_{\to_1}(A,\ B \to_2 C_i)(x)$.

Proof. Since $\to_1, \to_2$ satisfy (P14) and (P15), one has

$$\begin{aligned} & sust_{\to_1}(A,\ \sup_{i\in I} B_i \to_2 C)(x) = A(x) \to_1 (\sup_{i\in I} B_i(x) \to_2 C(x)) \\ & = A(x) \to_1 \inf_{i\in I}(B_i(x) \to_2 C(x)) = \inf_{i\in I}(A(x) \to_1 (B_i(x) \to_2 C(x))) \\ & = \inf_{i\in I} sust_{\to_1}(A,\ B_i \to_2 C)(x), \end{aligned}$$

and

$$\begin{aligned} & sust_{\to_1}(A,\ B \to_2 \inf_{i\in I} C_i)(x) = A(x) \to_1 (B(x) \to_2 \inf_{i\in I} C_i(x)) \\ & = A(x) \to_1 \inf_{i\in I}(B(x) \to_2 C_i(x)) = \inf_{i\in I}(A(x) \to_1 (B(x) \to_2 C_i(x))) \\ & = \inf_{i\in I} sust_{\to_1}(A,\ B \to_2 C_i)(x). \quad \square \end{aligned}$$

It is straightforward to derive Corollary 3.3 from Proposition 3.5.

Corollary 3.3. *Suppose that $\to$ is a fuzzy implication satisfying (P14) and (P15), then the following properties hold for any $A, B, C, B_i, C_i \in F(X)$ $(X, I \neq \varnothing,\ i \in I,\ x \in X)$:*
(i) $sust_\to(A,\ \sup_{i\in I} B_i \to C)(x) = \inf_{i\in I} sust_\to(A,\ B_i \to C)(x)$,
(ii) $sust_\to(A,\ B \to \inf_{i\in I} C_i)(x) = \inf_{i\in I} sust_\to(A,\ B \to C_i)(x)$.

Remark 3.2. It is noted that an R-implication is also a fuzzy implication satisfying (P7), (P14) and (P15). Therefore in Proposition 3.3, Proposition 3.4 and Proposition 3.5, as well as Corollary 3.3, if $\to$ is an R-implication (or $\to_1, \to_2$ are R-implications), then the corresponding conclusions are also correct.

We can get Corollary 3.4 from Proposition 3.4 and Proposition 2.5.

Corollary 3.4. *Suppose that* $\rightarrow$ *is an (S, N)-implication, then* $sust_{\rightarrow}(A,\ B \rightarrow C)(x) = sust_{\rightarrow}(B,\ A \rightarrow C)(x)$ *holds for any* $A, B, C \in F(X)$ $(X \neq \varnothing,\ x \in X)$.

In a similar way, we can get Proposition 3.6, Proposition 3.7 and Proposition 3.8, together with Corollary 3.5.

Proposition 3.6. *Suppose that* $\rightarrow$ *is a fuzzy implication satisfying (P14) and (P15), and that* P_i, Q *are fuzzy relations from* X *to* Y*, then the following properties hold* $(X, Y, I \neq \varnothing,\ i \in I,\ x \in X,\ y \in Y)$:
(i) $sust_{\rightarrow}(\sup_{i\in I} P_i, Q)(x, y) = \inf_{i\in I} sust_{\rightarrow}(P_i, Q)(x, y)$,
(ii) $sust_{\rightarrow}(Q, \inf_{i\in I} P_i)(x, y) = \inf_{i\in I} sust_{\rightarrow}(Q, P_i)(x, y)$.

Proposition 3.7. *Suppose that* $\rightarrow$ *is a fuzzy implication satisfying (P7), and that* P_1, P_2, P_3 *are fuzzy relations from* X *to* Y*, then the following property holds* $(X, Y, I \neq \varnothing,\ i \in I,\ x \in X,\ y \in Y)$:

$$sust_{\rightarrow}(P_1,\ P_2 \rightarrow P_3)(x, y) = sust_{\rightarrow}(P_2,\ P_1 \rightarrow P_3)(x, y).$$

Proposition 3.8. *Suppose that* $\rightarrow_1, \rightarrow_2$ *are two fuzzy implications satisfying (P14) and (P15), and that* P_1, P_2, Q_i *are fuzzy relations from* X *to* Y*, then the following properties hold* $(X, Y, I \neq \varnothing,\ i \in I,\ x \in X,\ y \in Y)$:
(i) $sust_{\rightarrow_1}(P_1,\ \sup_{i\in I} Q_i \rightarrow_2 P_2)(x, y) = \inf_{i\in I} sust_{\rightarrow_1}(P_1,\ Q_i \rightarrow_2 P_2)(x, y)$,
(ii) $sust_{\rightarrow_1}(P_1,\ P_2 \rightarrow_2 \inf_{i\in I} Q_i)(x, y) = \inf_{i\in I} sust_{\rightarrow_1}(P_1,\ P_2 \rightarrow_2 Q_i)(x, y)$.

Corollary 3.5. *Suppose that* $\rightarrow$ *is a fuzzy implication satisfying (P14) and (P15), and that* P_1, P_2, Q_i *are fuzzy relations from* X *to* Y*, then the following properties hold* $(X, Y, I \neq \varnothing,\ i \in I,\ x \in X,\ y \in Y)$:
(i) $sust_{\rightarrow}(P_1,\ \sup_{i\in I} Q_i \rightarrow P_2)(x, y) = \inf_{i\in I} sust_{\rightarrow}(P_1,\ Q_i \rightarrow P_2)(x, y)$,
(ii) $sust_{\rightarrow}(P_1,\ P_2 \rightarrow \inf_{i\in I} Q_i)(x, y) = \inf_{i\in I} sust_{\rightarrow}(P_1,\ P_2 \rightarrow Q_i)(x, y)$.

Remark 3.3. Similar to Remark 3.2, if $\rightarrow$ is an R-implication (or $\rightarrow_1, \rightarrow_2$ are R-implications), then the corresponding conclusions in Proposition 3.6, Proposition 3.7, Proposition 3.8 and Corollary 3.5 are also correct.

Corollary 3.6 is obtained from Proposition 3.7 and Proposition 2.5.

Corollary 3.6. *Suppose that* $\rightarrow$ *is an (S, N)-implication, and that* P_1, P_2, P_3 *are fuzzy relations from* X *to* Y*, then the following property holds* $(X, Y, I \neq \varnothing,\ i \in I,\ x \in X,\ y \in Y)$: $sust_{\rightarrow}(P_1,\ P_2 \rightarrow P_3)(x, y) = sust_{\rightarrow}(P_2,\ P_1 \rightarrow P_3)(x, y)$.

Proposition 3.9. *Let* $\rightarrow_1, \rightarrow_2$ *be two fuzzy implications satisfying (P3), and* P *any fuzzy relation from* X *to* Y, $Q_1(x, y) = C(x) \rightarrow_1 D_1(y)$, $Q_2(x, y) = C(x) \rightarrow_1 D_2(y)$ *(in which* $X, Y \neq \varnothing$ *and* $C \in <F(X), \leq_F>, D_1, D_2 \in <F(Y), \leq_F>$*). If* $D_1 \leq_F D_2$*, then* $sust_{\rightarrow_2}(P, Q_1)(x, y) \leq sust_{\rightarrow_2}(P, Q_2)(x, y)$ *holds for any* $x \in X,\ y \in Y$.

Proof. Since $D_1 \leq_F D_2$ and $\rightarrow_1, \rightarrow_2$ satisfy (P3), it follows that $C(x) \rightarrow_1 D_1(y) \leq C(x) \rightarrow_1 D_2(y)$ and $sust_{\rightarrow_2}(P, Q_1)(x, y) = P(x, y) \rightarrow_2 (C(x) \rightarrow_1 D_1(y)) \leq P(x, y) \rightarrow_2 (C(x) \rightarrow_1 D_2(y)) = sust_{\rightarrow_2}(P, Q_2)(x, y)$ hold for any $x \in X,\ y \in Y$. $\square$

Proposition 3.10 is proved in a similar way as Proposition 3.9.

Proposition 3.10. *Let* $\rightarrow_1, \rightarrow_2$ *be two fuzzy implications satisfying (P2) and (P3), and* P *any fuzzy relation from* X *to* Y, $Q_1(x, y) = C_1(x) \rightarrow_1 D(y)$, $Q_2(x, y) = C_2(x) \rightarrow_1 D(y)$ *(in which* $X, Y \neq \varnothing$ *and* $C_1, C_2 \in <F(X), \leq_F>, D \in <F(Y), \leq_F>$*). If* $C_2 \leq_F C_1$*, then* $sust_{\rightarrow_2}(P, Q_1)(x, y) \leq sust_{\rightarrow_2}(P, Q_2)(x, y)$ *holds for any* $x \in X,\ y \in Y$.

4. $\alpha(u, v)$-symmetric implicational method for FMP

Based on the two-dimensional sustaining degree, (8) can also be expressed as

$$sust_{\rightarrow_2}(Q, Q^*)(u, v) \geq \alpha(u, v). \tag{12}$$

Here $Q(u, v) = A(u) \rightarrow_1 B(v)$ and $Q^*(u, v) = A^*(u) \rightarrow_1 B^*(v)$.

For convenience, denote $a' = 1 - a$ for any $a \in [0, 1]$ and $A'(x) = 1 - A(x)$ for any fuzzy set A, meanwhile we also denote $T'(a, b) = 1 - T(a, b)$ for any mapping $T : [0, 1]^2 \rightarrow [0, 1]$.

For the FMP problem (1), from the viewpoint of the $\alpha(u, v)$-symmetric implicational method, we can obtain the following principle:

α(u,v)-symmetric implicational principle for FMP: The conclusion B^* of FMP problem (1) is the smallest fuzzy set satisfying (12) in $< F(V), \leq_F >$.

It is evident that such symmetric implicational principle for FMP improves the previous one discussed in [20].

Definition 4.1. Let $A, A^* \in F(U)$, $B \in F(V)$, if B^* (in $< F(V), \leq_F >$) makes (12) hold for any $u \in U, v \in V$, then B^* is called an α(u,v)-FMP-symmetric implicational solution.

Definition 4.2. Suppose that $A, A^* \in F(U)$, $B \in F(V)$, and that non-empty set $\mathbb{E}_{\alpha(u,v)}$ is the set of all α(u,v)-FMP-symmetric implicational solutions, and finally that D^* (in $< F(V), \leq_F >$) is the infimum of $\mathbb{E}_{\alpha(u,v)}$. Then D^* is called an α(u,v)-InfP-quasi symmetric implicational solution. And, if D^* is the minimum of $\mathbb{E}_{\alpha(u,v)}$, then D^* is also called an α(u,v)-MinP-symmetric implicational solution.

Proposition 4.1 results from Proposition 3.9.

Proposition 4.1. *If $\to_1, \to_2$ satisfy (P3), and D_1 is an α(u,v)-FMP-symmetric implicational solution, and $D_1 \leq_F D_2$ (where $D_1, D_2 \in < F(V), \leq_F >$). Then D_2 is an α(u,v)-FMP-symmetric implicational solution.*

Remark 4.1. Suppose that $\to_1, \to_2$ satisfy (P3). In Definition 4.2, A, A^*, B should be unchangeable and B^* changeable, while B^* should make (12) hold for any $u \in U, v \in V$. For (12), once there exists an α(u,v)-FMP-symmetric implicational solution B^*, then every fuzzy set D which is larger than B^* ($D \in F(V)$), will be an α(u,v)-FMP-symmetric implicational solution. This means that there are many α(u,v)-FMP-symmetric implicational solutions, including $B^*(v) \equiv 1$ ($v \in V$). This last is a special solution, for which (12) always holds no matter what major premise $A \to_1 B$ and minor premise A^* are adopted. Therefore, when the optimal α(u,v)-FMP-symmetric implicational solution exists, it should be the smallest one; in other words, it should be the infimum of all solutions (i.e. the infimum of $\mathbb{E}_{\alpha(u,v)}$).

Assume that the maximum of $sust_{\to_2}(Q, Q^*)(u, v)$ for FMP at every point (u, v) is $M(u, v)$.

Proposition 4.2. *([20]) If $\to_1, \to_2$ satisfy (P3), then $M(u, v) = (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 1)$. Especially, if $\to_1, \to_2$ also satisfy (P5), then $M(u, v) = 1$.*

To guarantee that (12) holds, it is necessary that $\alpha(u, v) \leq M(u, v)$ holds for any $u \in U, v \in V$.

It follows from Lemma 2.1 that $< F(V), \leq_F >$ is a complete lattice. Thus the α(u,v)-InfP-quasi symmetric implicational solution (i.e., the infimum of $\mathbb{E}_{\alpha(u,v)}$) exists since the non-empty set $\mathbb{E}_{\alpha(u,v)} \subset F(V)$.

Proposition 4.3. *If $\to_1, \to_2$ satisfy (P3) and (P10), then the α(u,v)-InfP-quasi symmetric implicational solution B^* is the α(u,v)-MinP-symmetric implicational solution.*

Proof. Note that $\mathbb{E}_{\alpha(u,v)} = \{D^* \in F(V) \mid (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 D^*(v)) \geq \alpha(u, v),\ \ u \in U, v \in V\}$, and that the α(u,v)-InfP-quasi symmetric implicational solution $B^* = \inf \mathbb{E}_{\alpha(u,v)}$. Assume on the contrary that $B^* \notin \mathbb{E}_{\alpha(u,v)}$, then there exist fuzzy sets $B_1, B_2, \cdots$ in $\mathbb{E}_{\alpha(u,v)}$ such that

$$\lim_{n\to\infty} B_n(v) = B^*(v), \qquad v \in V. \tag{13}$$

Since $B_1, B_2, \cdots \in \mathbb{E}_{\alpha(u,v)}$, we get ($n = 1, 2, \cdots,\ u \in U,\ \ v \in V$):

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B_n(v)) \geq \alpha(u, v). \tag{14}$$

Because $B^* = \inf \mathbb{E}_{\alpha(u,v)}$, we obtain $B_n(v) \geq B^*(v)$ ($v \in V,\ \ n = 1, 2, \cdots$), and then it follows from (13) that $B^*(v)$ is the right limit of $\{B_n(v) \mid n = 1, 2, \cdots\}$ ($v \in V$). Noting that the fuzzy implications $\to_1, \to_2$ satisfies (P3) and (P10), thus it follows from (14) that ($u \in U, v \in V$):

$$\begin{aligned}\alpha(u, v) &\leq \lim_{n\to\infty} \{(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B_n(v))\} = (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \\ &= sust_{\to_2}(Q, Q^*)(u, v).\end{aligned}$$

Hence $B^* \in \mathbb{E}_{\alpha(u,v)}$, which is a contradiction.

As a result, $B^* \in \mathbb{E}_{\alpha(u,v)}$ and hence B^* is the minimum of $\mathbb{E}_{\alpha(u,v)}$. Therefore, B^* is the α(u,v)-MinP-symmetric implicational solution. □

It follows from Proposition 4.3 and Proposition 2.2 that we can get Theorem 4.1.

Theorem 4.1. *If $\to_1, \to_2$ are R-implications, then the α(u,v)-InfP-quasi symmetric implicational solution B^* is the α(u,v)-MinP-symmetric implicational solution.*

Theorem 4.2. *If $\to_1, \to_2$ are R-implications, and T_1, T_2 are respectively the mappings residual to $\to_1, \to_2$, then the α(u,v)-MinP-symmetric implicational solution can be expressed as follows:*

$$B^*(v) = \sup_{u \in U}\{T_1(A^*(u), T_2(A(u) \to_1 B(v),\ \alpha(u,v)))\},\ \ v \in V. \tag{15}$$

Proof. It follows from (15) that

$$T_1(A^*(u), T_2(A(u) \to_1 B(v),\ \alpha(u,v))) \le B^*(v),\ u \in U, v \in V.$$

Because $(T_1, \to_1)$, $(T_2, \to_2)$ are two residual pairs, we have $T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \le A^*(u) \to_1 B^*(v)$ and $\alpha(u,v) \le sust_{\to_2}(Q, Q^*)(u,v)$ $(u \in U, v \in V)$, i.e., B^* satisfies (12) for all $u \in U, v \in V$. Hence B^* expressed by (15) belongs to $\mathbb{E}_{\alpha(u,v)}$.

Assume that $D \in \langle F(V), \le_F\rangle$, and that

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 D(v)) \ge \alpha(u,v),\ u \in U, v \in V.$$

Noting that $(T_1, \to_1)$ and $(T_2, \to_2)$ are two residual pairs, we obtain $T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \le A^*(u) \to_1 D(v)$, and then

$$T_1(A^*(u), T_2(A(u) \to_1 B(v),\ \alpha(u,v))) \le D(v),\ u \in U, v \in V.$$

So $D(v)$ is an upper bound of

$$\{T_1(A^*(u), T_2(A(u) \to_1 B(v),\ \alpha(u,v))) \mid u \in U\},\ \ v \in V.$$

Hence it follows from (15) that $B^* \le_F D$. These mean that B^* is the minimum of $\mathbb{E}_{\alpha(u,v)}$.

Therefore, it follows from Definition 4.2 that B^* expressed by (15) is the α(u,v)-MinP-symmetric implicational solution. $\square$

Example 4.1. The following fuzzy implications are all R-implications, which include Lukasiewicz implication I_{LK}, Gödel implication I_{GD}, Goguen implication I_{GG}, Fodor implication I_{FD} ([22,30]), and I_{EP} (which is residual to the t-norm of Einstein product defined as $T_{EP}(x,y) = xy/[2-(x+y-xy)]$), I_{YG} (which is residual to the t-norm of Yager defined as $T_{YG-\omega}(x,y) = 1 - \min[1, ((1-x)^{\omega} + (1-y)^{\omega})^{1/\omega}]$, where ω is equal to 0.5) [31].

$$I_{LK}(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 1-x+y & \text{if } x > y \end{cases}, \qquad I_{GD}(a,b) = \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } x > y \end{cases},$$

$$I_{GG}(x,y) = \begin{cases} 1 & \text{if } x \le y \\ y/x & \text{if } x > y \end{cases}, \qquad I_{FD}(a,b) = \begin{cases} 1 & \text{if } x \le y \\ (1-x) \vee y & \text{if } x > y \end{cases},$$

$$I_{EP}(x,y) = \begin{cases} 1 & \text{if } x \le y \\ (2y-xy)/(x+y-xy) & \text{if } x > y \end{cases}, \qquad I_{YG}(a,b) = \begin{cases} 1 & \text{if } x \le y \\ 1-(\sqrt{1-y}-\sqrt{1-x})^2 & \text{if } x > y \end{cases}.$$

Proposition 4.4 can be then easily formulated.

Proposition 4.4. *The t-norm corresponding to the R-implications $I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{EP}, I_{YG}$ in residual pairs are as follows, respectively.*

$$T_{LK}(x,y) = \begin{cases} x+y-1 & \text{if } x+y > 1 \\ 0 & \text{if } x+y \le 1 \end{cases}, \qquad T_{GD}(x,y) = x \wedge y, \qquad T_{GG}(x,y) = x \times y,$$

$$T_{FD}(x,y) = \begin{cases} x \wedge y & \text{if } x+y > 1 \\ 0 & \text{if } x+y \le 1 \end{cases}, \qquad T_{EP}(x,y) = xy/(2-x-y+xy),$$

$$T_{YG}(x,y) = \begin{cases} 1-(k(x,y))^2 & \text{if } k(x,y) \le 1 \\ 0 & \text{if } k(x,y) > 1 \end{cases}, \text{ where } k(x,y) = \sqrt{1-x} + \sqrt{1-y}.$$

In a similar way, we obtain the following theorems.

Theorem 4.3. *If $\to_1, \to_2$ are (S, N)-implications satisfying (P10), and T_1, T_2 are respectively the mappings residual to $\to_1, \to_2$, then the α(u,v)-MinP-symmetric implicational solution can be expressed as (15).*

Theorem 4.4. *If $\to_1$ is an R-implication, and $\to_2$ an (S, N)-implication satisfying (P10), and T_1, T_2 are respectively the mappings residual to $\to_1, \to_2$, then the α(u,v)-MinP-symmetric implicational solution can be expressed as (15).*

Table 1
Examples of basic (S, N)-implications $I_{S,N}$.

S	N	$I_{S,N}$
S_M	N_C	I_{KD}
S_P	N_C	I_{RC}
S_{nM}	N_C	I_{FD}
S_{LK}	N_C	I_{LK}
S_M	N_K	I_{MK}
Any S	N_{D1}	I_D
Any S	N_{D2}	I_{TD}

Theorem 4.5. *If $\rightarrow_1$ is an (S, N)-implication satisfying (P10), and $\rightarrow_2$ an R-implication, and T_1, T_2 are respectively the mappings residual to $\rightarrow_1, \rightarrow_2$, then the α(u,v)-MinP-symmetric implicational solution can be expressed as (15).*

It is obvious to obtain Lemma 4.1.

Lemma 4.1. *For the S-implication $I_{S,N}$, if the t-conorm S is right-continuous, then $I_{S,N}$ satisfies (P10).*

The basic t-conorms include the following ($a, b \in [0, 1]$):
(i) Maximum: $S_M(a, b) = \max(a, b)$,
(ii) Probabilistic sum: $S_P(a, b) = a + b - ab$,
(iii) Lukasiewicz: $S_{LK}(a, b) = \min(a + b, 1)$,
(iv) Nilpotent maximum: $S_{nM}(a, b) = \begin{cases} 1, & a + b \geq 1 \\ \max(a, b), & \text{otherwise} \end{cases}$,
(v) Drastic sum: $S_{DR}(a, b) = \begin{cases} 1, & a, b \in (0, 1] \\ \max(a, b), & \text{otherwise} \end{cases}$.

It is not difficult to find that S_M, S_P, S_{LK}, S_{nM} are right-continuous, while S_{DR} is not. As for S_{DR}, the main value is always constant (when $a, b \in (0, 1]$), thus S_{DR} is not appropriate in light of the requirements imposed on fuzzy inference, therefore in this paper we do not consider S_{DR} any more.

The examples of basic (S, N)-implications $I_{S,N}$ are shown in Table 1, where the related fuzzy implications are Kleene–Dienes implication I_{KD}, Reichenbach implication I_{RC}, I_{MK}, the least (S, N)-implication I_D, and the largest (S, N)-implication I_{TD}.

$$I_{KD}(x, y) = (1 - x) \vee y, \qquad I_{RC}(x, y) = 1 - x + xy, \qquad I_{MK}(x, y) = (1 - x^2) \vee y,$$
$$I_D(x, y) = \begin{cases} 1 & \text{if } x = 0 \\ y & \text{if } x > 0 \end{cases}, \qquad I_{TD}(x, y) = \begin{cases} 1 & \text{if } x < 1 \\ y & \text{if } x = 1 \end{cases}.$$

Proposition 4.5. *The mappings corresponding to the (S, N)-implications $I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}$ in residual pairs are as follows, respectively.*

$$T_{KD}(x, y) = \begin{cases} y & \text{if } x + y > 1 \\ 0 & \text{if } x + y \leq 1 \end{cases}, \qquad T_{RC}(x, y) = \begin{cases} (x + y - 1)/x & \text{if } x + y > 1 \\ 0 & \text{if } x + y \leq 1 \end{cases},$$
$$T_{MK}(x, y) = \begin{cases} y & \text{if } 1 - x^2 < y \\ 0 & \text{if } 1 - x^2 \geq y \end{cases}, \qquad T_D(x, y) = \begin{cases} y & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases},$$
$$T_{TD}(x, y) = \begin{cases} y & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases}.$$

Proof. Here it is easy to find that $I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}$ are all fuzzy implications satisfying (P10).

We only prove I_D (note that $I_D(x, y) = \begin{cases} 1 & \text{if } x = 0 \\ y & \text{if } x > 0 \end{cases}$). We achieve from Proposition 2.9 that

$$\begin{aligned} T_D(x, y) &= \inf\{u \in [0, 1] \mid y \leq I_D(x, u)\} \\ &= \inf(\{u \in [0, 1] \mid x = 0, y \leq I_D(x, u)\} \cup \{u \in [0, 1] \mid x > 0, y \leq I_D(x, u)\}) \\ &= [\inf\{u \in [0, 1] \mid x = 0, y \leq 1\}] \wedge [\inf\{u \in [0, 1] \mid x > 0, y \leq u\}]. \end{aligned}$$

If $x = 0$, then $T_D(x, y) = 0 \wedge [\wedge\varnothing] = 0 \wedge 1 = 0$.

If $x > 0$, then $T_D(x, y) = [\wedge\varnothing] \wedge y = 1 \wedge y = y$.

Thus, it follows that $T_D(x, y) = \begin{cases} y & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$. □

From the results mentioned above, $I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{EP}, I_{YG}$ are all R-implications, and $I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}$ are all fuzzy implications satisfying (P10). Then it follows from Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 4.5 that we derive Proposition 4.6.

Proposition 4.6. *If* $\rightarrow_1, \rightarrow_2 \in \{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{EP}, I_{YG}, I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}\}$*, then the α(u,v)-MinP-symmetric implicational solution can be expressed by* $B^*(v) = \sup_{u\in U}\{T_1(A^*(u), T_2(A(u) \rightarrow_1 B(v),\ \alpha(u,v)))\},\ v \in V$.

Example 4.2. Here we show two specific examples of the α(u,v)-MinP-symmetric implicational solution.

(i) Let $\rightarrow_1$ take I_{GD} and $\rightarrow_2$ take I_{GG}, then it follows from Proposition 4.6 that the α(u,v)-MinP-symmetric implicational solution is as follows:

$$\begin{aligned} B^*(v) &= \sup_{u\in U}\{T_{GD}(A^*(u), T_{GG}(I_{GD}(A(u), B(v)),\ \alpha(u,v)))\} \\ &= \sup_{u\in U}\{A^*(u) \wedge (I_{GD}(A(u), B(v)) \times \alpha(u,v))\}, \qquad v \in V. \end{aligned}$$

(ii) Let $\rightarrow_1$ take I_{LK} and $\rightarrow_2$ take I_{FD}. If $I_{LK}(A(u), B(v)) + \alpha(u,v) > 1,\ \ A^*(u) + I_{LK}(A(u), B(v)) > 1$ and $A^*(u) + \alpha(u,v) > 1$ hold, then

$$T_{LK}(A^*(u), T_{FD}(I_{LK}(A(u), B(v)), \alpha(u,v))) = A^*(u) + (I_{LK}(A(u), B(v)) \wedge \alpha(u,v)) - 1.$$

Otherwise, we have $T_{LK}(A^*(u), T_{FD}(I_{LK}(A(u), B(v)), \alpha(u,v))) = 0$.

Denote

$$\begin{aligned} E_v &= \{u \in U \mid I_{LK}(A(u), B(v)) + \alpha(u,v) > 1,\ A^*(u) + I_{LK}(A(u), B(v)) > 1,\ A^*(u) + \alpha(u,v) > 1\} \\ &= \{u \in U \mid A^*(u) \wedge I_{LK}(A(u), B(v)) > 1 - \alpha(u,v),\ A^*(u) + I_{LK}(A(u), B(v)) > 1\}, \end{aligned}$$

then the α(u,v)-MinP-symmetric implicational solution is as follows:

$$\begin{aligned} B^*(v) &= \sup_{u\in U}\{T_{LK}(A^*(u), T_{FD}(I_{LK}(A(u), B(v)), \alpha(u,v)))\} \\ &= \sup_{u\in E_v}\{T_{LK}(A^*(u), T_{FD}(I_{LK}(A(u), B(v)), \alpha(u,v)))\} \vee \sup_{u\in U-E_v}\{T_{LK}(A^*(u), T_{FD}(I_{LK}(A(u), B(v)), \alpha(u,v)))\} \\ &= \sup_{u\in E_v}\{T_{LK}(A^*(u), T_{FD}(I_{LK}(A(u), B(v)), \alpha(u,v)))\} \vee \sup_{u\in U-E_v}\{0\} \\ &= \sup_{u\in E_v}\{A^*(u) + (I_{LK}(A(u), B(v)) \wedge \alpha(u,v)) - 1\}, \qquad v \in V. \end{aligned}$$

5. $\alpha(u, v)$-symmetric implicational method for FMT

Aiming at the FMT problem (2), from the viewpoint of the $\alpha(u,v)$-symmetric implicational method, we establish the following principle (which obviously improves the previous one presented in [20]):

α(u,v)-symmetric implicational principle for FMT: The conclusion A^* of FMT problem (2) is the largest fuzzy set satisfying (12) in $< F(U), \leq_F >$.

Definition 5.1. Let $A \in F(U),\ B, B^* \in F(V)$, if A^* (in $< F(U), \leq_F >$) makes (12) hold for any $u \in U, v \in V$, then A^* is called an α(u,v)-FMT-symmetric implicational solution.

Definition 5.2. Suppose that $A \in F(U),\ B, B^* \in F(V)$, and that non-empty set $\mathbb{F}_{\alpha(u,v)}$ is the set of all α(u,v)-FMT-symmetric implicational solutions, and finally that C^* (in $< F(U), \leq_F >$) is the supremum of $\mathbb{F}_{\alpha(u,v)}$. Then C^* is called an α(u,v)-SupT-quasi symmetric implicational solution. And, if C^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$, then C^* is also called an α(u,v)-MaxT-symmetric implicational solution.

Proposition 5.1 results from Proposition 3.10.

Proposition 5.1. *If* $\rightarrow_1, \rightarrow_2$ *satisfy (P2) and (P3), and* C_1 *is an α(u,v)-FMT-symmetric implicational solution, and* $C_2 \leq_F C_1$ *(where* $C_1, C_2 \in < F(U), \leq_F >$*), then* C_2 *is an α(u,v)-FMT-symmetric implicational solution.*

Remark 5.1. Suppose that $\rightarrow_1, \rightarrow_2$ satisfy (P2) and (P3). For Definition 5.2, A, B, B^* should be fixed, and A^* should make (12) hold for any $u \in U, v \in V$. As for (12), if there exists an α(u,v)-FMT-symmetric implicational solution A^*, then every fuzzy set C which is smaller than A^* ($C \in F(U)$), will also be an α(u,v)-FMT-symmetric implicational solution. So there exist many α(u,v)-FMT-symmetric implicational solutions, which include $A^*(u) \equiv 0$ $(u \in U)$. This last is a specific one, for which (12) always holds no matter what $A \to_1 B$ and B^* are chosen. As a result, if the optimal α(u,v)-FMT-symmetric implicational solution exists, then it should be the largest one or the supremum of $\mathbb{F}_{\alpha(u,v)}$.

Assume that the maximum of $sust_{\to_2}(Q, Q^*)(u, v)$ for FMT at every point (u, v) is $L(u, v)$.

Proposition 5.2. *If $\to_1$ satisfies (P2) and $\to_2$ satisfies (P3), then $L(u, v) = (A(u) \to_1 B(v)) \to_2 (0 \to_1 B^*(v))$. Especially, if $\to_1$ satisfies (P4) and $\to_2$ satisfies (P5), then $L(u, v) = 1$.*

Proof. Take $A^*(u) \equiv 0$ $(u \in U)$, then (12) is equal to $(A(u) \to_1 B(v)) \to_2 (0 \to_1 B^*(v))$.

Conversely, since $\to_1$ satisfies (P2), it follows that $A^*(u) \to_1 B^*(v) \le 0 \to_1 B^*(v)$ (noting that $0 \le A^*(u),\ u \in U$). Noting that $\to_1$ satisfies (P3), we further have

$$sust_{\to_2}(Q, Q^*)(u, v) \le (A(u) \to_1 B(v)) \to_2 (0 \to_1 B^*(v))$$

holds for any $u \in U, v \in V$. Furthermore, if $\to_1$ satisfies (P4) and $\to_2$ satisfies (P5), it is obvious to get $L(u, v) = 1$. □

To guarantee (12) holds, it is necessary that $\alpha(u, v) \le L(u, v)$ holds for any $u \in U, v \in V$. We get from Lemma 2.1 that $< F(U), \le_F >$ is a complete lattice. So the α(u,v)-SupT-quasi symmetric implicational solution (i.e., the supremum of $\mathbb{F}_{\alpha(u,v)}$) uniquely exists because the non-empty set $\mathbb{F}_{\alpha(u,v)} \subset F(U)$.

Proposition 5.3. *If $\to_1$ satisfies (P2) and (P9), and $\to_2$ satisfies (P10), then the α(u,v)-SupT-quasi symmetric implicational solution A^* is the α(u,v)-MaxT-symmetric implicational solution.*

Proof. It is noted that the α(u,v)-SupT-quasi symmetric implicational solution $A^* = \sup \mathbb{F}_{\alpha(u,v)}$, it is enough to prove that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. Notice that

$$\mathbb{F}_{\alpha(u,v)} = \{C^* \in F(U) \mid (A(u) \to_1 B(v)) \to_2 (C^*(u) \to_1 B^*(v)) \ge \alpha(u, v),\ u \in U, v \in V\}.$$

Assume to the contrary that $A^* \notin \mathbb{F}_{\alpha(u,v)}$. Then there exist fuzzy sets $A_1, A_2, \cdots$ in $\mathbb{F}_{\alpha(u,v)}$ such that

$$\lim_{n\to\infty} A_n(u) = A^*(u), \quad u \in U. \tag{16}$$

Because $A^* = \sup \mathbb{F}_{\alpha(u,v)}$, we obtain $A_n(u) \le A^*(u)$ $(u \in U,\ n = 1, 2, \cdots)$, and then it follows from (16) that $A^*(u)$ is the left limit of $\{A_n(u) \mid n = 1, 2, \cdots\}$ $(u \in U)$. Thus we get (noting that $\to_1$ satisfies (P9))

$$\lim_{n\to\infty} \{A_n(u) \to_1 B^*(v)\} = A^*(u) \to_1 B^*(v), \quad u \in U,\ v \in V. \tag{17}$$

Since the fuzzy implication $\to_1$ satisfies (P2), we have $A_n(u) \to_1 B^*(v) \ge A^*(u) \to_1 B^*(v)$ $(u \in U, v \in V,\ n = 1, 2, \cdots)$, and then it follows from (17) that $A^*(u) \to_1 B^*(v)$ is the right limit of $\{A_n(u) \to_1 B^*(v) \mid n = 1, 2, \cdots\}$ $(u \in U, v \in V)$.

Taking into account that $A_1, A_2, \cdots \in \mathbb{F}_{\alpha(u,v)}$, we have $(n = 1, 2, \cdots)$:

$$(A(u) \to_1 B(v)) \to_2 (A_n(u) \to_1 B^*(v)) \ge \alpha(u, v), \quad u \in U,\ v \in V.$$

Thus we obtain (by noting that $\to_2$ satisfies (P10)):

$$\begin{aligned}\alpha(u, v) &\le \lim_{n\to\infty} \{(A(u) \to_1 B(v)) \to_2 (A_n(u) \to_1 B^*(v))\} \\ &= (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \\ &= sust_{\to_2}(Q, Q^*)(u, v), \quad u \in U, v \in V,\end{aligned}$$

which contradicts the assumption. As a result, we achieve $A^* \in \mathbb{F}_{\alpha(u,v)}$, and hence A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. □

It follows from Proposition 5.3 and Proposition 2.2 that we can get Theorem 5.1.

Theorem 5.1. *If $\to_2$ is an R-implication, then the α(u,v)-SupT-quasi symmetric implicational solution A^* is the α(u,v)-MaxT-symmetric implicational solution.*

Theorem 5.2. *If $\to_1, \to_2$ are R-implications, and T_2 is the mapping residual to $\to_2$, then the α(u,v)-MaxT-symmetric implicational solution can be expressed as follows:*

$$A^*(u) = \inf_{v\in V} \{T_2(A(u) \to_1 B(v),\ \alpha(u, v)) \to_1 B^*(v)\}, \quad u \in U. \tag{18}$$

Proof. Since $\to_1$ is an R-implication, then it follows from Proposition 2.3 that $\to_1$ satisfies (P12).

We get from (18) that

$$A^*(u) \leq T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \to_1 B^*(v),\ u \in U, v \in V.$$

Because $\to_1$ satisfies (P12) and $(T_2, \to_2)$ is a residual pair, we obtain

$$T_2(A(u) \to_1 B(v), \alpha(u,v)) \leq A^*(u) \to_1 B^*(v)$$

and then $\alpha(u,v) \leq sust_{\to_2}(Q, Q^*)(u,v)$ $(u \in U, v \in V)$, that is, (12) holds for any $u \in U, v \in V$. Hence A^* expressed as (18) belongs to $\mathbb{F}_{\alpha(u,v)}$.

Furthermore, we verify that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. Assume that $C \in < F(U), \leq_F >$, and that

$$(A(u) \to_1 B(v)) \to_2 (C(u) \to_1 B^*(v)) \geq \alpha(u,v),\ u \in U, v \in V.$$

Since $(T_2, \to_2)$ is a residual pair and $\to_1$ satisfies (P12), we have $T_2(A(u) \to_1 B(v), \alpha(u,v)) \leq C(u) \to_1 B^*(v)$ and $C(u) \leq T_2(A(u) \to_1 B(v), \alpha(u,v)) \to_1 B^*(v)$ $(u \in U, v \in V)$. So $C(u)$ is a lower bound of

$$\{T_2(A(u) \to_1 B(v), \alpha(u,v)) \to_1 B^*(v) \mid v \in V\},\ \ u \in U.$$

Hence it follows from (18) that $C \leq_F A^*$. Consequently, A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$.

According to Definition 5.2, we achieve that A^* expressed as (18) is the α(u,v)-MaxT-symmetric implicational solution. □

In a similar way as for Theorem 5.2, we can obtain Theorem 5.3.

Theorem 5.3. *Suppose that $\to_1$ is an R-implication, and that $\to_2$ takes the (S, N)-implication I_{S_1,N_1} satisfying (P10), and finally that T_2 is the mappings residual to $\to_2$. Then the α(u,v)-MaxT-symmetric implicational solution can be computed as (18).*

Since $I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{EP}, I_{YG}$ are all R-implications, and $I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}$ are all fuzzy implications satisfying (P10), it follows from Theorem 5.2 and Theorem 5.3 that we can have Proposition 5.4.

Proposition 5.4. *Suppose that $\to_1 \in \{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{EP}, I_{YG}\}$, and that $\to_2 \in \{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{EP}, I_{YG}, I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}\}$, then the α(u,v)-MaxT-symmetric implicational solution is $A^*(u) = \inf_{v \in V}\{T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \to_1 B^*(v)\},\ u \in U$.*

Theorem 5.4. *Suppose that $\to_1, \to_2$ respectively take the (S, N)-implication I_{S_1,N_1}, I_{S_2,N_2} satisfying (P10), and that T_1, T_2 are respectively the mappings residual to $\to_1, \to_2$. Then the α(u,v)-MaxT-symmetric implicational solution can be computed as follows:*

$$A^*(u) = \inf_{v \in V}\{N_1(T_1(N_1(B^*(v)), T_2(A(u) \to_1 B(v),\ \alpha(u,v))))\},\ \ u \in U. \tag{19}$$

Proof. Note that $\to_1, \to_2$ are two (S, N)-implications satisfying (P10), then it follows from Proposition 2.9 that there exist residual pairs $(T_1, \to_1), (T_2, \to_2)$. Since $\to_1$ employs the (S, N)-implication I_{S_1,N_1}, it follows from Proposition 2.5 that $\to_1$ satisfies (P17), i.e., has the law of contraposition w.r.t. the strong negation N_1.

It follows from (19) that

$$A^*(u) \leq N_1(T_1(N_1(B^*(v)), T_2(A(u) \to_1 B(v),\ \alpha(u,v)))),\ u \in U, v \in V.$$

Because $\to_1$ satisfies the law of contraposition w.r.t. N_1, and $(T_1, \to_1), (T_2, \to_2)$ are two residual pairs, we obtain $(u \in U, v \in V)$

$$T_1(N_1(B^*(v)), T_2(A(u) \to_1 B(v),\ \alpha(u,v))) \leq N_1(A^*(u)),$$

$$T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \leq N_1(B^*(v)) \to_1 N_1(A^*(u)),$$

$$T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \leq A^*(u) \to_1 B^*(v)$$

$$\alpha(u,v) \leq (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) = sust_{\to_2}(Q, Q^*)(u,v).$$

That is, (12) holds for any $u \in U, v \in V$, which implies that A^* expressed as (19) belongs to $\mathbb{F}_{\alpha(u,v)}$.

Moreover, we prove that A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$. Assume that $C \in < F(U), \leq_F >$, and that

$$(A(u) \to_1 B(v)) \to_2 (C(u) \to_1 B^*(v)) \geq \alpha(u,v),\ u \in U, v \in V.$$

Considering that $(T_1, \to_1), (T_2, \to_2)$ are two residual pairs and that $\to_1$ has the law of contraposition w.r.t. N_1, we have $(u \in U, v \in V)$

$$T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \le C(u) \to_1 B^*(v),$$

$$T_2(A(u) \to_1 B(v),\ \alpha(u,v)) \le N_1(B^*(v)) \to_1 N_1(C(u)),$$

$$T_1(N_1(B^*(v)), T_2(A(u) \to_1 B(v), \alpha(u,v))) \le N_1(C(u)),$$

$$C(u) \le N_1(T_1(N_1(B^*(v)), T_2(A(u) \to_1 B(v), \alpha(u,v)))).$$

Thus $C(u)$ is a lower bound of

$$\{N_1(T_1(N_1(B^*(v)), T_2(A(u) \to_1 B(v), \alpha(u,v)))) \mid v \in V\},\quad u \in U.$$

So it follows from (19) that $C \le_F A^*$. As a result, A^* is the maximum of $\mathbb{F}_{\alpha(u,v)}$.

By virtue of Definition 5.2, we obtain that A^* expressed as (19) is the α(u,v)-MaxT-symmetric implicational solution. □

As in case of Theorem 5.4, we derive Theorem 5.5.

Theorem 5.5. *Suppose that $\to_1$ takes the (S, N)-implication I_{S_1,N_1} satisfying (P10), and that $\to_2$ is an R-implication, and finally that T_1, T_2 are respectively the mappings residual to $\to_1, \to_2$. Then the α(u,v)-MaxT-symmetric implicational solution can be computed as (19).*

From Theorem 5.4 and Theorem 5.5 we obtain Proposition 5.5.

Proposition 5.5. *Suppose that $\to_1 \in \{I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}\}$ and $\to_2 \in \{I_{LK}, I_{GD}, I_{GG}, I_{FD}, I_{EP}, I_{YG}, I_{KD}, I_{RC}, I_{MK}, I_D, I_{TD}\}$, then the α(u,v)-MaxT-symmetric implicational solution is $A^*(u) = \inf_{v\in V}\{N_1(T_1(N_1(B^*(v)), T_2(A(u) \to_1 B(v),\ \alpha(u,v))))\}$, $u \in U$.*

Example 5.1. Here we show two specific examples of the α(u,v)-MaxT-symmetric implicational solution.

(i) Let $\to_1$ take I_{LK} and $\to_2$ take I_{GG}, then it follows from Proposition 5.4 that the α(u,v)-MaxT-symmetric implicational solution is as follows:

$$\begin{aligned} A^*(u) &= \inf_{v\in V}\{I_{LK}(T_{GG}(I_{LK}(A(u), B(v)),\ \alpha(u,v)),\ B^*(v))\} \\ &= \inf_{v\in V}\{I_{LK}((I_{LK}(A(u), B(v)) \times \alpha(u,v)),\ B^*(v))\},\quad u \in U. \end{aligned}$$

If $I_{LK}(A(u), B(v)) \times \alpha(u,v) > B^*(v)$, then

$$I_{LK}((I_{LK}(A(u), B(v)) \times \alpha(u,v)),\ B^*(v)) = 1 - (I_{LK}(A(u), B(v)) \times \alpha(u,v)) + B^*(v).$$

Otherwise, we have $I_{LK}((I_{LK}(A(u), B(v)) \times \alpha(u,v)),\ B^*(v)) = 1$.

Denote

$$F_u = \{v \in V \mid I_{LK}(A(u), B(v)) \times \alpha(u,v) > B^*(v)\},$$

then the α(u,v)-MaxT-symmetric implicational solution is as follows:

$$\begin{aligned} A^*(u) &= \inf_{v\in F_u}\{I_{LK}((I_{LK}(A(u), B(v)) \times \alpha(u,v)),\ B^*(v))\} \\ &\quad \wedge \inf_{v\in V-F_u}\{I_{LK}((I_{LK}(A(u), B(v)) \times \alpha(u,v)),\ B^*(v))\} \\ &= \inf_{v\in F_u}\{1 - (I_{LK}(A(u), B(v)) \times \alpha(u,v)) + B^*(v)\} \wedge \inf_{v\in V-F_u}\{1\} \\ &= \inf_{v\in F_u}\{1 - (I_{LK}(A(u), B(v)) \times \alpha(u,v)) + B^*(v)\},\quad u \in U. \end{aligned}$$

(ii) Let $\to_1$ take I_{KD} and $\to_2$ take I_{GD}, then it follows from Proposition 5.5 that the α(u,v)-MaxT-symmetric implicational solution is as follows:

$$\begin{aligned} A^*(u) &= \inf_{v\in V}\{N_1(T_{KD}(N_1(B^*(v)), T_{GD}[I_{KD}(A(u), B(v)),\ \alpha(u,v)]))\} \\ &= \inf_{v\in V}\{1 - (T_{KD}[(1 - B^*(v)), [((1 - A(u)) \vee B(v)) \wedge \alpha(u,v)]])\},\quad u \in U. \end{aligned}$$

If $(1 - B^*(v)) + ([(1 - A(u)) \vee B(v)] \wedge \alpha(u,v)) > 1$ hold (i.e., $((1 - A(u)) \vee B(v)) \wedge \alpha(u,v) > B^*(v)$), then

$$T_{KD}((1 - B^*(v)), (((1 - A(u)) \vee B(v)) \wedge \alpha(u,v))) = ((1 - A(u)) \vee B(v)) \wedge \alpha(u,v).$$

Otherwise, we have $T_{KD}((1 - B^*(v)), (((1 - A(u)) \vee B(v)) \wedge \alpha(u,v))) = 0$.

Denote

$$F_u = \{v \in V \mid ((1 - A(u)) \vee B(v)) \wedge \alpha(u,v) > B^*(v)\},$$

then the α(u,v)-MaxT-symmetric implicational solution is as follows:

$$\begin{aligned} A^*(u) = &\inf_{v \in F_u} \{1 - (T_{KD}((1 - B^*(v)), (((1 - A(u)) \vee B(v)) \wedge \alpha(u,v))))\} \\ &\wedge \inf_{v \in V - F_u} \{1 - (T_{KD}((1 - B^*(v)), (((1 - A(u)) \vee B(v)) \wedge \alpha(u,v))))\} \\ = &\inf_{v \in F_u} \{1 - (((1 - A(u)) \vee B(v)) \wedge \alpha(u,v))\} \wedge \inf_{v \in V - F_u} \{1 - 0\} \\ = &\inf_{v \in F_u} \{1 - (((1 - A(u)) \vee B(v)) \wedge \alpha(u,v))\}, \quad u \in U. \end{aligned}$$

6. Examples

Here we provide four illustrative examples (including two continuous cases and two discrete ones) to deal with the α(u,v)-symmetric implicational method.

With n rules, (2) and (3) come as:

$$\text{FMP: from n rules } A_i \to B_i \text{ and } A^*, \ \text{ compute } B^*, \tag{20}$$

$$\text{FMT: from n rules } A_i \to B_i \text{ and } B^*, \ \text{ compute } A^*. \tag{21}$$

The overall inference rule is frequently chosen to be $\phi(u,v) \triangleq \vee_{i=1}^n (A_i(u) \to_1 B_i(v))$.

Consequently, (12) should be expressed as:

$$[\vee_{i=1}^n (A_i(u) \to_1 B_i(v))] \to_2 (A^*(u) \to_1 B^*(v)) \geq \alpha(u,v), \tag{22}$$

or

$$sust_{\to_2}(\phi, Q^*)(u,v) \geq \alpha(u,v). \tag{23}$$

Suppose that the α(u,v)-MinP-symmetric implicational solution (or the α(u,v)-MaxT-symmetric implicational solution) from (12) is $\Psi(A(u) \to_1 B(v))$, then it is easy to obtain that the α(u,v)-MinP-symmetric implicational solution (or the α(u,v)-MaxT-symmetric implicational solution) derived from (23) is $\Psi(\phi(u,v))$.

Example 6.1. Let $U = V = [0,1]$, $A(u) = (2+u)/4$, $B(v) = (3-2v)/4$, $A^*(u) = (2-u)/2$ and $\alpha(u,v) = (3+v-u)/4$ (in which $u, v \in [0,1]$). Suppose that $\to_1 = I_{GD}$, $\to_2 = I_{KD}$ in the $\alpha(u,v)$-symmetric implicational method for FMP. We now calculate the α(u,v)-MinP-symmetric implicational solution from Theorem 4.4.

To begin with, we have

$$I_{GD}(A(u), B(v)) = \begin{cases} 1, & \text{if } \frac{2+u}{4} \leq \frac{3-2v}{4}, \\ \frac{3-2v}{4}, & \text{if } \frac{2+u}{4} > \frac{3-2v}{4} \end{cases} = \begin{cases} 1, & \text{if } u + 2v \leq 1 \\ \frac{3-2v}{4}, & \text{if } u + 2v > 1. \end{cases}.$$

Then, it follows from Theorem 4.4 that the α(u,v)-MinP-symmetric implicational solution is as follows ($v \in V$),

$$\begin{aligned} B^*(v) &= \sup_{u \in U} \{T_1(A^*(u), T_2(A(u) \to_1 B(v), \ \alpha(u,v)))\} \\ &= \sup_{u \in U} \{T_{GD}(A^*(u), T_{KD}(I_{GD}(A(u), B(v)), \ \alpha(u,v)))\} \\ &= \sup_{u \in U} \{A^*(u) \wedge T_{KD}(I_{GD}(A(u), B(v)), \ \alpha(u,v))\} \\ &= \sup_{u \in [0,1]} \left\{ \frac{2-u}{2} \wedge T_{KD}(1, \frac{3+v-u}{4}) \;\middle|\; u + 2v \leq 1 \right\} \vee \sup_{u \in [0,1]} \left\{ \frac{2-u}{2} \wedge T_{KD}(\frac{3-2v}{4}, \frac{3+v-u}{4}) \;\middle|\; u + 2v > 1 \right\} \\ &= \sup_{u \in [0,1]} \left\{ \frac{2-u}{2} \wedge \frac{3+v-u}{4} \;\middle|\; u + 2v \leq 1 \right\} \vee \sup_{u \in [0,1]} \left\{ \frac{2-u}{2} \wedge \frac{3+v-u}{4} \;\middle|\; u + 2v > 1, \ \frac{3-2v}{4} + \frac{3+v-u}{4} > 1 \right\} \\ &= \sup_{u \in [0,1]} \left\{ \frac{2-u}{2} \wedge \frac{3+v-u}{4} \;\middle|\; u + 2v \leq 1 \right\} \vee \sup_{u \in [0,1]} \left\{ \frac{2-u}{2} \wedge \frac{3+v-u}{4} \;\middle|\; u > 1 - 2v, \ u < 2 - v \right\}. \end{aligned}$$

(i) Suppose that $0 \leq v \leq 1/2$. Then $0 \in \{u \in [0,1], \ u + 2v \leq 1\}$. It is easy to find $2 - v > 1 - 2v$, then $\{ u \in [0,1], u > 1 - 2v, \ u < 2 - v\} \neq \varnothing$. Considering that $\frac{2-u}{2} \wedge \frac{3+v-u}{4}$ is non-increasing w.r.t. u, we get

$$B^*(v) = (1 \wedge \frac{3+v}{4}) \vee (\frac{2-(1-2v)}{2} \wedge \frac{3+v-(1-2v)}{4}) = \frac{3+v}{4} \vee (\frac{2+4v}{4} \wedge \frac{2+3v}{4}) = \frac{3+v}{4} \vee \frac{2+3v}{4}$$
$$= \frac{3+v}{4}.$$

Here $\frac{3+v}{4} \geq \frac{2+3v}{4}$ is from $v \leq 1/2$.

(ii) Suppose that $1 \geq v > 1/2$. Then $\{u \in [0,1],\ u+2v \leq 1\} = \varnothing$. Here $0 \in \{\ u \in [0,1], u > 1-2v,\ u < 2-v\}$, taking into account that $\frac{2-u}{2} \wedge \frac{3+v-u}{4}$ is non-increasing w.r.t. u, we have

$$B^*(v) = (\sup \varnothing) \vee (1 \wedge \frac{3+v}{4}) = 0 \vee \frac{3+v}{4} = \frac{3+v}{4}.$$

Together we obtain that the α(u,v)-MinP-symmetric implicational solution is

$$B^*(v) = \frac{3+v}{4}. \quad \square$$

Example 6.2. Let $U = V = [0,1]$, $A(u) = (2+u)/4$, $B(v) = (3-2v)/4$, $B^*(v) = (1-v)/2$ and $\alpha(u,v) = (3+u-v)/4$ (in which $u, v \in [0,1]$). Suppose that $\rightarrow_1 = I_{LK}$, $\rightarrow_2 = I_{FD}$ in the $\alpha(u,v)$-symmetric implicational method for FMT. We now calculate the α(u,v)-MaxT-symmetric implicational solution from Theorem 5.2.

To begin with, we have

$$I_{LK}(A(u), B(v)) = \begin{cases} 1, & \text{if } \frac{2+u}{4} \leq \frac{3-2v}{4}, \\ 1 - \frac{2+u}{4} + \frac{3-2v}{4}, & \text{if } \frac{2+u}{4} > \frac{3-2v}{4} \end{cases} = \begin{cases} 1, & \text{if } u+2v \leq 1 \\ \frac{5-u-2v}{4}, & \text{if } u+2v > 1. \end{cases}$$

Then, it follows from Theorem 5.2 that the α(u,v)-MaxT-symmetric implicational solution is as follows ($u \in U$),

$$\begin{aligned}
A^*(u) &= \inf_{v\in V}\{T_2(A(u) \rightarrow_1 B(v),\ \alpha(u,v)) \rightarrow_1 B^*(v)\} \\
&= \inf_{v\in V}\{I_{LK}(T_{FD}(I_{LK}(A(u), B(v)),\ \alpha(u,v)), B^*(v))\} \\
&= \inf_{v\in[0,1]}\left\{I_{LK}(T_{FD}(1,\ \frac{3+u-v}{4}), \frac{1-v}{2}) \,\middle|\, u+2v \leq 1\right\} \\
&\quad \wedge \inf_{v\in[0,1]}\left\{I_{LK}(T_{FD}(\frac{5-u-2v}{4},\ \frac{3+u-v}{4}), \frac{1-v}{2}) \,\middle|\, u+2v > 1\right\} \\
&= \inf_{v\in[0,1]}\left\{I_{LK}(\frac{3+u-v}{4}, \frac{1-v}{2}) \,\middle|\, u+2v \leq 1\right\} \wedge \inf_{v\in[0,1]}\left\{I_{LK}(\frac{5-u-2v}{4} \wedge \frac{3+u-v}{4}, \frac{1-v}{2}) \,\middle|\, u+2v > 1\right\} \\
&= \inf_{v\in[0,1]}\left\{I_{LK}(\frac{3+u-v}{4}, \frac{1-v}{2}) \,\middle|\, u+2v \leq 1\right\} \wedge \inf_{v\in[0,1]}\left\{I_{LK}(\frac{5-u-2v}{4}, \frac{1-v}{2}) \,\middle|\, 2u+v \geq 2,\ u+2v > 1\right\} \\
&\quad \wedge \inf_{v\in[0,1]}\left\{I_{LK}(\frac{3+u-v}{4}, \frac{1-v}{2}) \,\middle|\, 2u+v < 2,\ u+2v > 1\right\} \\
&= \inf_{v\in[0,1]}\left\{1 - \frac{3+u-v}{4} + \frac{1-v}{2} \,\middle|\, u+2v \leq 1\right\} \wedge \inf_{v\in[0,1]}\left\{1 - \frac{5-u-2v}{4} + \frac{1-v}{2} \,\middle|\, 2u+v \geq 2,\ u+2v > 1\right\} \\
&\quad \wedge \inf_{v\in[0,1]}\left\{1 - \frac{3+u-v}{4} + \frac{1-v}{2} \,\middle|\, 2u+v < 2,\ u+2v > 1\right\} \\
&= \inf_{v\in[0,1]}\left\{\frac{3-u-v}{4} \,\middle|\, v \leq \frac{1-u}{2}\right\} \wedge \inf_{v\in[0,1]}\left\{\frac{1+u}{4} \,\middle|\, v \geq 2-2u,\ v > \frac{1-u}{2}\right\} \\
&\quad \wedge \inf_{v\in[0,1]}\left\{\frac{3-u-v}{4} \,\middle|\, v < 2-2u,\ v > \frac{1-u}{2}\right\}.
\end{aligned}$$

Here $T_{FD}(\frac{5-u-2v}{4},\ \frac{3+u-v}{4}) = \frac{5-u-2v}{4} \wedge \frac{3+u-v}{4}$ due to $\frac{5-u-2v}{4} + \frac{3+u-v}{4} = \frac{8-3v}{4} > 1$.

$I_{LK}(\frac{3+u-v}{4}, \frac{1-v}{2}) = 1 - \frac{3+u-v}{4} + \frac{1-v}{2}$ due to $\frac{3+u-v}{4} \geq \frac{1-v}{2}$.

$I_{LK}(\frac{5-u-2v}{4}, \frac{1-v}{2}) = 1 - \frac{5-u-2v}{4} + \frac{1-v}{2}$ due to $\frac{5-u-2v}{4} \geq \frac{1-v}{2}$.

(i) Suppose that $0 \le u < 1/2$. Then $\{ v \in [0,1], v \ge 2-2u, v > \frac{1-u}{2}\} = \varnothing$. Considering that $\frac{3-u-v}{4}$ are non-increasing w.r.t. v, then we have

$$A^*(u) = \frac{3-u-\frac{1-u}{2}}{4} \wedge \inf\varnothing \wedge \frac{3-u-(2-2u)}{4} = \frac{5-u}{8} \wedge 1 \wedge \frac{1+u}{4} = \frac{1+u}{4}.$$

Note that $\frac{5-u}{8} \ge \frac{1+u}{4}$ due to $u \le 1$.

(ii) Suppose that $1/2 \le u < 1$. Then $\{ v \in [0,1], v \ge 2-2u, v > \frac{1-u}{2}\} \ne \varnothing$. Considering that $\frac{3-u-v}{4}$ are non-increasing w.r.t. v, then we have

$$A^*(u) = \frac{3-u-\frac{1-u}{2}}{4} \wedge \frac{1+u}{4} \wedge \frac{3-u-(2-2u)}{4} = \frac{5-u}{8} \wedge \frac{1+u}{4} \wedge \frac{1+u}{4} = \frac{1+u}{4}.$$

(iii) Suppose that $u=1$. Then $\{ v \in [0,1], v \ge 2-2u, v > \frac{1-u}{2}\} \ne \varnothing$ and $\{ v \in [0,1], v < 2-2u,\ v > \frac{1-u}{2}\} = \varnothing$. Considering that $\frac{3-u-v}{4}$ are non-increasing w.r.t. v, then we have

$$A^*(u) = \frac{3-u-\frac{1-u}{2}}{4} \wedge \frac{1+u}{4} \wedge \inf\varnothing = \frac{5-u}{8} \wedge \frac{1+u}{4} \wedge 1 = \frac{1+u}{4}.$$

Together we have

$$A^*(u) = \frac{1+u}{4}, \quad u \in U. \quad \square$$

Example 6.3. Let $U = \{u_1, u_2, \cdots, u_5\}$ in which $u_1 = 0.2$, $u_2 = 0.4$, $u_3 = 0.6$, $u_4 = 0.8$, $u_5 = 1.0$, and $V = \{v_1\}$ where $v_1 = 0.6$, and $\alpha(u,v) = (1-u^2+v)/2$. Moreover, six rules $A_i \to B_i$ and the input A^* are as follows:

$$\begin{aligned}
A_1 &= \frac{0.8}{u_1} + \frac{0.3}{u_2} + \frac{0.2}{u_3} + \frac{0.3}{u_4} + \frac{0.6}{u_5},\ B_1 = \frac{0.2}{v_1},\\
A_2 &= \frac{0.9}{u_1} + \frac{0.4}{u_2} + \frac{0.0}{u_3} + \frac{0.5}{u_4} + \frac{1.0}{u_5},\ B_2 = \frac{0.2}{v_1},\\
A_3 &= \frac{0.5}{u_1} + \frac{0.7}{u_2} + \frac{0.8}{u_3} + \frac{0.7}{u_4} + \frac{0.5}{u_5},\ B_3 = \frac{0.5}{v_1},\\
A_4 &= \frac{0.4}{u_1} + \frac{0.6}{u_2} + \frac{0.3}{u_3} + \frac{0.6}{u_4} + \frac{0.4}{u_5},\ B_4 = \frac{0.5}{v_1},\\
A_5 &= \frac{0.2}{u_1} + \frac{0.9}{u_2} + \frac{0.7}{u_3} + \frac{1.0}{u_4} + \frac{0.6}{u_5},\ B_5 = \frac{0.8}{v_1},\\
A_6 &= \frac{0.3}{u_1} + \frac{1.0}{u_2} + \frac{0.6}{u_3} + \frac{0.9}{u_4} + \frac{0.7}{u_5},\ B_6 = \frac{0.8}{v_1},\\
A^* &= \frac{0.6}{u_1} + \frac{0.5}{u_2} + \frac{0.7}{u_3} + \frac{0.8}{u_4} + \frac{0.3}{u_5}.
\end{aligned}$$

This is an example for fuzzy classification based on fuzzy expert system, where 3 classes correspond to $B(v_1) = 0.2$, $B(v_1) = 0.5$, $B(v_1) = 0.8$. Suppose that $\to_1 = I_{GD}$, $\to_2 = I_{KD}$ in the $\alpha(u,v)$-symmetric implicational method for FMP. Then we present processing required to develop the α(u,v)-MinP-symmetric implicational solution B^* to determine which class B^* belongs to.

For $v_1 = 0.6$, we get

$$\begin{aligned}
&\phi(u_1, v_1) = \vee_{i=1}^{6}(A_i(u_1) \to_1 B_i(v_1))\\
&= (0.8 \to_1 0.2) \vee (0.9 \to_1 0.2) \vee (0.5 \to_1 0.5) \vee (0.4 \to_1 0.5) \vee (0.2 \to_1 0.8) \vee (0.3 \to_1 0.8)\\
&= 0.2 \vee 0.2 \vee 1.0 \vee 1.0 \vee 1.0 \vee 1.0 = 1.0.
\end{aligned}$$

Similarly, we can get $\phi(u_2, v_1) = 0.8$, $\phi(u_3, v_1) = 1.0$, $\phi(u_4, v_1) = 0.8$, $\phi(u_5, v_1) = 1.0$.

From Theorem 4.4, we can get the α(u,v)-MinP-symmetric implicational solution is as follows:

$$\begin{aligned}
&B^*(v_1) = \sup_{u \in U}\{T_1(A^*(u), T_2(\phi(u,v),\ \alpha(u,v)))\}\\
&= \sup_{u \in U}\{T_{GD}(A^*(u), T_{KD}(\phi(u,v),\ \alpha(u,v)))\}\\
&= [T_{GD}(A^*(u_1), T_{KD}(\phi(u_1,v_1),\ \alpha(u_1,v_1)))] \vee [T_{GD}(A^*(u_2), T_{KD}(\phi(u_2,v_1),\ \alpha(u_2,v_1)))]\\
&\quad \vee \cdots \vee [T_{GD}(A^*(u_5), T_{KD}(\phi(u_5,v_1),\ \alpha(u_5,v_1)))]
\end{aligned}$$

$$= [T_{GD}(0.6, T_{KD}(1.0,\ \alpha(0.2, 0.6)))] \vee [T_{GD}(0.5, T_{KD}(0.8,\ \alpha(0.4, 0.6)))] \vee [T_{GD}(0.7, T_{KD}(1.0,\ \alpha(0.6, 0.6)))]$$
$$\vee [T_{GD}(0.8, T_{KD}(0.8,\ \alpha(0.8, 0.6)))] \vee [T_{GD}(0.3, T_{KD}(1.0,\ \alpha(1.0, 0.6)))]$$
$$= [T_{GD}(0.6, T_{KD}(1.0,\ 0.78))] \vee [T_{GD}(0.5, T_{KD}(0.8,\ 0.72))] \vee [T_{GD}(0.7, T_{KD}(1.0,\ 0.62))]$$
$$\vee [T_{GD}(0.8, T_{KD}(0.8,\ 0.48))] \vee [T_{GD}(0.3, T_{KD}(1.0,\ 0.3))]$$
$$= [T_{GD}(0.6, 0.78)] \vee [T_{GD}(0.5, 0.72)] \vee [T_{GD}(0.7, 0.62)] \vee [T_{GD}(0.8, 0.48)] \vee [T_{GD}(0.3, 0.3)]$$
$$= 0.6 \vee 0.5 \vee 0.62 \vee 0.48 \vee 0.3 = 0.62.$$

Since 0.62 is closest to 0.5, the classification result is the second class. □

Example 6.4. Let $\alpha(u, v) = (1 + u - v)/2$, and $U = \{u_1\}$ where $u_1 = 0.6$, and $V = \{v_1, v_2, v_3, v_4\}$ where $v_1 = 0.2$, $v_2 = 0.4$, $v_3 = 0.6$, $v_4 = 0.8$. The rules and input are as follows:

$$A_1 = \frac{0.3}{u_1},\ B_1 = \frac{0.2}{v_1} + \frac{0.3}{v_2} + \frac{0.1}{v_3} + \frac{0.5}{v_4},$$
$$A_2 = \frac{0.3}{u_1},\ B_2 = \frac{0.1}{v_1} + \frac{0.8}{v_2} + \frac{0.0}{v_3} + \frac{0.7}{v_4},$$
$$A_3 = \frac{0.6}{u_1},\ B_3 = \frac{0.5}{v_1} + \frac{0.7}{v_2} + \frac{0.4}{v_3} + \frac{0.0}{v_4},$$
$$A_4 = \frac{0.6}{u_1},\ B_4 = \frac{0.3}{v_1} + \frac{0.6}{v_2} + \frac{0.3}{v_3} + \frac{0.8}{v_4},$$
$$A_5 = \frac{0.9}{u_1},\ B_5 = \frac{0.6}{v_1} + \frac{1.0}{v_2} + \frac{0.7}{v_3} + \frac{0.1}{v_4},$$
$$A_6 = \frac{0.9}{u_1},\ B_6 = \frac{0.7}{v_1} + \frac{0.8}{v_2} + \frac{0.6}{v_3} + \frac{0.9}{v_4},$$
$$B^* = \frac{0.5}{v_1} + \frac{0.9}{v_2} + \frac{0.8}{v_3} + \frac{0.2}{v_4}.$$

This is an example for fuzzy classification based on fuzzy expert system, in which three classes correspond to $A(u_1) = 0.3$, $A(u_1) = 0.6$, $A(u_1) = 0.9$. Suppose that $\to_1 = I_{LK}$, $\to_2 = I_{FD}$ in the $\alpha(u, v)$-symmetric implicational method for FMT. We now calculate the α(u,v)-MaxT-symmetric implicational solution.

For $u_1 = 0.6$, we get

$$\phi(u_1, v_1) = \vee_{i=1}^{6}(A_i(u_1) \to_1 B_i(v_1))$$
$$= (0.3 \to_1 0.2) \vee (0.3 \to_1 0.1) \vee (0.6 \to_1 0.5) \vee (0.6 \to_1 0.3) \vee (0.9 \to_1 0.6) \vee (0.9 \to_1 0.7)$$
$$= 0.9 \vee 0.8 \vee 0.9 \vee 0.7 \vee 0.7 \vee 0.8 = 0.9.$$

Similarly, we can get $\phi(u_1, v_2) = 1.0$, $\phi(u_1, v_3) = 0.8$, $\phi(u_1, v_4) = 1.0$.

It follows from Theorem 5.2 that the α(u,v)-MaxT-symmetric implicational solution is as follows:

$$A^*(u_1) = \inf_{v \in V}\{T_2(\phi(u, v), \alpha(u, v)) \to_1 B^*(v)\}$$
$$= \inf_{v \in V}\{I_{LK}(T_{FD}(\phi(u, v), \alpha(u, v)),\ B^*(v))\}$$
$$= [I_{LK}(T_{FD}(\phi(u_1, v_1), \alpha(u_1, v_1)),\ B^*(v_1))] \wedge [I_{LK}(T_{FD}(\phi(u_1, v_2), \alpha(u_1, v_2)),\ B^*(v_2))]$$
$$\wedge [I_{LK}(T_{FD}(\phi(u_1, v_3), \alpha(u_1, v_3)),\ B^*(v_3))] \wedge [I_{LK}(T_{FD}(\phi(u_1, v_4), \alpha(u_1, v_4)),\ B^*(v_4))]$$
$$= [I_{LK}(T_{FD}(0.9, \alpha(0.6, 0.2)),\ 0.5)] \wedge [I_{LK}(T_{FD}(1.0, \alpha(0.6, 0.4)),\ 0.9)]$$
$$\wedge [I_{LK}(T_{FD}(0.8, \alpha(0.6, 0.6)),\ 0.8)] \wedge [I_{LK}(T_{FD}(1.0, \alpha(0.6, 0.8)),\ 0.2)]$$
$$= [I_{LK}(T_{FD}(0.9, 0.7),\ 0.5)] \wedge [I_{LK}(T_{FD}(1.0, 0.6),\ 0.9)]$$
$$\wedge [I_{LK}(T_{FD}(0.8, 0.5),\ 0.8)] \wedge [I_{LK}(T_{FD}(1.0, 0.4),\ 0.2)]$$
$$= [I_{LK}(0.7, 0.5)] \wedge [I_{LK}(0.6, 0.9)] \wedge [I_{LK}(0.5, 0.8)] \wedge [I_{LK}(0.4, 0.2)]$$
$$= 0.8 \wedge 1.0 \wedge 1.0 \wedge 0.8 = 0.8.$$

Since 0.8 is nearest to 0.9, the third class is what is required. □

7. Discussion

Here we include some discussion related to the $\alpha(u,v)$-symmetric implicational method.

i) If $\alpha(u,v) \equiv \alpha$ $(u \in U,\ v \in V)$, then the $\alpha(u,v)$-symmetric implicational method degenerates into the α-symmetric implicational method.

ii) If $\alpha(u,v) = M(u,v)$ $(u \in U,\ v \in V)$, (12) is transformed into

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \geq M(u,v).$$

Since the maximum of (12) for FMP at every point (u,v) is $M(u,v)$, we have $(u \in U, v \in V)$

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \leq M(u,v).$$

Hence we achieve $(u \in U, v \in V)$

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) = M(u,v).$$

Consequently the $\alpha(u,v)$-symmetric implicational method for FMP degenerates into the symmetric implicational method for FMP.

Similarly, if $\alpha(u,v) = L(u,v)$ $(u \in U,\ v \in V)$, then the $\alpha(u,v)$-symmetric implicational method for FMT degenerates into the symmetric implicational method for FMT.

iii) There are three reasons to generalize α to $\alpha(u,v)$.

(a) As mentioned in Section 1, the solutions to the basic symmetric implicational method are coming from $(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)) \geq W(u,v)$. When $W(u,v)$ is a constant, then the basic symmetric implicational method is a special case of the α-symmetric implicational method (i.e., $W(u,v) = \alpha$ for any $u \in U, v \in V$). However, when $W(u,v)$ is not a constant, then there is no direct relationship between the α-symmetric implicational method and the basic symmetric implicational method. As a result, the previous theory of symmetric implicational method is not perfect. Focusing on such problem, by generalizing α to $\alpha(u,v)$, the $\alpha(u,v)$-symmetric implicational method contains the α-symmetric implicational method and the basic symmetric implicational method as its particular cases, leading to that these symmetric implicational methods form a unified view.

(b) The basic symmetric implicational method already implies the idea of $W(u,v)$. It is noted that $W(u,v)$ is the maximum of (6) at (u,v), which is a point-to-point value for any $u \in U, v \in V$. Obviously, $\alpha(u,v)$ can also be regarded as a generalization of $W(u,v)$. Consequently, it is natural to use $\alpha(u,v)$ to realize the symmetric implicational method of fuzzy inference.

(c) Since $\alpha(u,v)$ is a generalization of α and $W(u,v)$, using $\alpha(u,v)$, provides a more exact representation and offers a more flexible mechanism for the symmetric implicational inference idea. As a result, the $\alpha(u,v)$-symmetric implicational method exhibits some theoretical generalization in contrast with the α-symmetric implicational method and the basic symmetric implicational method.

iv) If $\to_1, \to_2$ employ the same fuzzy implication, then the $\alpha(u,v)$-symmetric implicational method degenerates into the corresponding case of the full implication method. Specially, the $\alpha(u,v)$-symmetric implicational method degenerates into the α-full implication method when $\to_1 = \to_2$ and $\alpha(u,v) \equiv \alpha$ $(u \in U,\ v \in V)$. And the $\alpha(u,v)$-symmetric implicational method degenerates into the full implication method when $\to_1 = \to_2$ and $\alpha(u,v) = M(u,v)$ or $L(u,v)$ $(u \in U,\ v \in V)$. These kinds of full implication methods are all particular cases of the $\alpha(u,v)$-symmetric implicational method.

For p fuzzy implications in the inference framework, any kind of full implication method can get p kinds of specific reasoning forms. However, the $\alpha(u,v)$-symmetric implicational method can provide p^2 kinds, which include the p kinds derived from the full implication method. For example, in this work, 11 specific fuzzy implications are included. Then the full implication method can get 11 reasoning forms, while the $\alpha(u,v)$-symmetric implicational method can provide $11 * 11 = 121$ reasoning forms. As a result, the $\alpha(u,v)$-symmetric implicational method can achieve more forms of fuzzy inference.

8. Conclusions

The symmetric implicational method with two-dimensional sustaining degree (i.e., the $\alpha(u,v)$-symmetric implicational method) is proposed and investigated. The main contributions and conclusions are outlined as follows.

(i) The sustaining degree is generalized to the two-dimensional sustaining degree, and some properties of such two kinds of sustaining degrees are carefully discussed.

(ii) The $\alpha(u,v)$-symmetric implicational methods for FMP and FMT are researched, including the following three aspects:

(a) New symmetric implicational principles are brought forward, which improve the previous ones.

(b) Unified forms of the $\alpha(u,v)$-symmetric implicational method are attained for FMP and FMT, in which $\to_1, \to_2$ employs an R-implication or (S, N)-implication.

(c) The optimal solutions of the $\alpha(u,v)$-symmetric implicational method are obtained for several specific fuzzy implications.

(iii) We show four computing examples including two continuous ones and two discrete ones.

(iv) We provide some discussions for the $\alpha(u, v)$-symmetric implicational method and related methods. It is noted that the proposed method lets all symmetric implicational methods compose a united entirety.

This study could deliver further improvements to the areas of fuzzy inference and fuzzy controllers. In the future, it is worth investigating the $\alpha(u, v)$-symmetric implicational method and other fuzzy inference strategies from the viewpoint of granular computing (see [32–34]), and generalizing the existing constructs to granular fuzzy inference schemes.

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