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Symmetric implicational algorithm derived from intuitionistic fuzzy entropy

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Abstract

On account of the idea of maximum fuzzy entropy and symmetric implicational mechanism under the environment of intuitionistic fuzzy sets, we come up with the intuitionistic fuzzy entropy derived symmetric implicational (IFESI) algorithm. Above all, novel symmetric implicational principles are presented, and the unified solutions of the IFESI algorithm are acquired for IFMP (intuitionistic fuzzy modus ponens) and IFMT (intuitionistic fuzzy modus ponens), which build upon in view of residual intuitionistic implications. Thereafter, the reductive properties and continuity of the IFESI algorithm are validated for IFMP and IFMT. In addition, the IFESI algorithm is extended to the α-IFESI algorithm, and the unified solutions of the α-IFESI algorithm are obtained for IFMP and IFMT. Finally, two examples of fuzzy classification for the α-IFESI algorithm are presented to demonstrate the detailed computing process of the IFESI algorithm.

Keywords: Fuzzy reasoning, intuitionistic fuzzy entropy, compositional rule of inference, symmetric implicational algorithm, reductive property, continuity.

1 Introduction

The research of fuzzy set [23, 26, 33] has been carried out by scholars all over the world, and has made rapid development in just a few decades. Fuzzy set theory has many important directions, it is worth mentioning that fuzzy reasoning [1, 25], as an indispensable part of the theory, is a top priority of research. There are two kinds of reasoning processes in fuzzy sets, namely fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT):

$$\text{FMP: from given rule } A \to B, \text{and input } A^*, \text{calculate } B^*(\text{output}), \tag{1}$$

$$\text{FMP: from given rule } A \to B, \text{and input } B^*, \text{calculate } A^*(\text{output}), \tag{2}$$

Here A and A^* are fuzzy sets on domain X, while B and B^* are fuzzy sets on domain Y.

To deal with the FMP and FMT problems, the compositional rule of inference (CRI) was proposed by Zadeh in 1973 [2, 34]. Fuzzy reasoning by using CRI method has been applied in many fields. Enlightened by the idea of the CRI method, Wang proposed a new algorithm called the full implication method in 1999 [29].

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The core idea of the full implication method is to find the smallest $B^* \in F(Y)$ for FMP (or largest $A^* \in F(X)$ for FMT) so that the following equation

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)), \tag{3}$$

is maximized for all $x \in X$, $y \in Y$, where $F(X)$ and $F(Y)$ represent the sets of all fuzzy sets on X and Y, respectively.

Since then, many scholars have carried out studies on the properties and improvements of the full implication method, and they have made many valuable contributions to the important improvements of the full implication method. Wang solved the formalization problem of the full implication method for general fuzzy reasoning [30]. Wang and Fu presented the unified form of the solution of the full implication method based on regular implication [32]. Pei [20] systematically discussed the full implication method with unified forms. Zhang and Yang [35] provided the concept of generalized roots of theories, and then investigated the full implication method in four kinds of propositional logic systems. Pei discussed the same problem based on the first-order logic system, and proposed a perfect unified full implication method in logic reasoning system [21]. Wang and Duan came up with a finer measure for appraising robustness, and researched the robustness of the full implication method related to the finer measurements [31]. Luo and Liu [15] put forward the sensitivity interval-valued fuzzy connectives, and then discussed the robustness of interval-valued full implication method. Luo and Wang [16] proposed general reasoning algorithms on the basis of interval-valued fuzzy sets, that was the interval-valued full implication method based on the left-continuous t-representable t-norm. Based on such researches, it is found that the full implication method has a strict logical basis, reducibility, continuity, robustness and many other advantages.

From a quite different viewpoint, the first and third fuzzy implications in (3) of the full implication method can be deemed as the implication connective in a logic system; and the second fuzzy implication in (3) reflects "if-then" relation of fuzzy inference model "if A implies B, then A^* implies B^*". In consideration of such idea, in [27], we generalized (3) into the following structure:

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_1 B^*(v)). \tag{4}$$

Here $\to_1, \to_2$ are two fuzzy implications. The corresponding fuzzy reasoning method is referred to as the symmetric implicational method. It was validated in [27] that the symmetric implicational method has achieved remarkable results. In [24], we further researched the $\alpha(u, v)$-symmetric implicational method, in which the R- and (S, N)-implications were used. Dai [8] established the predicate formal representation of the symmetric implication method in the formal logic system, which provided a solid logical foundation for the symmetric implication method.

However, there are many phenomena in the real world that cannot be expressed formally by fuzzy sets [28]. As a generalization of fuzzy sets, intuitionistic fuzzy sets introduce the concepts of membership and non-membership, which have a better ability to express the fuzziness of everyday things. Here we show an example. Let us analyze a traveling salesman who has a limit on how far he can travel and cannot arrive at all cities, but he has some information about the cities where the biggest sales can occur. As a result, the goal here is to maximize entire sales within a limited travel distance. Suppose that A is a collection of all the cities that the salesman can reach, and that $x \in A$ denotes a city reached by the salesman, then the membership degree of the cities reached by the salesman can be characterized by $\pi_A(x)$. On this occasion, an ordinary fuzzy set is able to be utilized, however it cannot represent the case where we need to appraise the quantity of cities that a salesman cannot reach. Aiming at such situation, we demand the intuitionistic fuzzy set, in which the degree of non-membership can be computed as $\tau_A(x)$. In addition, it can occur that the salesman reaches a city but is unable to sell his product because the customers are unavailable or the store is closed, which can be indicated by the hesitancy degree (as $1 - \pi_A(x) - \tau_A(x)$) of the intuitionistic fuzzy set. Furthermore, some operators, such as the modal operators, can be designed for intuitionistic fuzzy sets, but not for fuzzy sets. These operators exhibit a detailed evaluation for the existing information. Intuitionistic fuzzy set theory proposed by Atanassov has been widely used in cluster analysis, pattern recognition and group decision making [7], [14]. When the objects involved by fuzzy reasoning needs to be characterized by the intuitionistic fuzzy sets, fuzzy reasoning evolves into intuitionistic fuzzy reasoning, which is similar to ordinary fuzzy reasoning in its research methods. The core problem of intuitionistic fuzzy reasoning is to solve the IFMP (intuitionistic FMP) and IFMT (intuitionistic FMT) problem. Therefore, it is reasonable to include fuzzy reasoning into intuitionistic fuzzy reasoning. In recent years, some scholars have tried to construct the logical basis of intuitionistic fuzzy reasoning. Deschrijver [9, 10] completed a preliminary study of the relevant theories of intuitionistic fuzzy implication, and put forward the intuitionistic fuzzy triangle norm and triangle conorm. In [17], the interval-valued fuzzy reasoning method based on similarity measure for FMP and FMT was proposed, and the solutions of interval-valued fuzzy reasoning method based on similarity measure were given. Zheng et al. put forward residual intuitionistic implication, then established the full implication method based on intuitionistic fuzzy sets [36].

In addition, there are usually more than a single fuzzy set that satisfies the maximum of (3), but only the minimum is taken as the inference result for FMP. Similarly, there are usually more than one fuzzy set that satisfies the maximum

(3), but only the maximum is taken as the inference result for FMT. Are they the most reasonable solutions? Aiming at such kinds of problems, Jaynes proposed the maximum entropy principle in 1975 to deal with uncertainty information problems. The central idea is that "in the process of reasoning with incomplete information, we must use the maximum distribution of entropy under the constraints of known conditions, which is the only unbiased dispatch we can do, and any other dispatch is equivalent to any assumption of our unknown information". Burillo and Bustince [6] presented the intuitionistic fuzzy entropy and gave the axiomatic definition and the calculation formula of intuitionistic fuzzy entropy. Szmidt and Kacprzyk [22] proposed an entropy measure with a geometric interpretation of intuitionistic fuzzy sets. To solve the problem of group decision making in which the criteria weight and the weight of decision makers were completely unknown in the intuitionistic fuzzy environment, Melo-Pinto et al. [19] proposed a multi-criteria group decision method based on improved intuitionistic fuzzy entropy and information integration operator. From the point of view of this maximum entropy principle, we found that the criterion of maximum entropy can offer an excellent interpretation for how to select the optimal outcome of the fuzzy reasoning strategy.

As a result, in this study, we combine the symmetric implicational idea with maximum entropy under the environment of intuitionistic fuzzy sets, and we propose the intuitionistic fuzzy entropy derived symmetric implicational (IFESI) algorithm, and then we extend it to the α-version.

The innovation points of the IFESI algorithm are reflected by the following aspects. First of all, the maximum entropy principle is introduced into the symmetric implicational idea, and new fuzzy reasoning principles are presented. Second, unified structures of optimal solutions of the IFESI algorithm are established for IFMP, IFMT together with the case of α-version. Finally, the reductive properties and continuity are validated for both IFMP and IFMT under the category of the IFESI algorithm.

2 Preliminaries

Definition 2.1. [5] *A fuzzy implication on $L = [0, 1]$ is a function $\rightarrow$ from L^2 to L such that the following conditions hold:*

$(P1)$ $0 \rightarrow 0 = 1$, $1 \rightarrow 1 = 1$, $1 \rightarrow 0 = 0$,
$(P2)$ $x \rightarrow z \geq y \rightarrow z$ *if* $x \leq y$,
$(P3)$ $x \rightarrow y \geq x \rightarrow z$ *if* $y \geq z$.

According to Definition 2.1, the relationship
$(P4)$ $0 \rightarrow x = x \rightarrow 1 = 1$ $(x \in [0, 1])$
holds for any fuzzy implication $\rightarrow$ (noting $0 \rightarrow 1 = 1$ obviously holds).

Definition 2.2. [13] *Suppose that $\otimes, \rightarrow$ are two $L^2 \rightarrow L$ mappings, $(\otimes, \rightarrow)$ is said to be a residual pair (or $\otimes$ and $\rightarrow$ are residual to each other), if $x \otimes y \leq z \iff y \leq x \rightarrow z$ holds for any $x, y, z \in [0, 1]$, which is called the residual condition.*

Definition 2.3. [13] *A fuzzy negation is a decreasing function $N : [0, 1] \rightarrow [0, 1]$ which satisfies $N(0) = 1$, $N(1) = 0$.*

For example, $N(x) = 1 - x$ is the classical negation.

Definition 2.4. [13] *A binary operation $\otimes$ is called the triangular norm (t-norm for short) if $\otimes$ satisfies the following four properties $(x, y, z \in L)$:*

$(P5)$ $x \otimes y = y \otimes x$,
$(P6)$ $(x \otimes y) \otimes z = x \otimes (y \otimes z)$,
$(P7)$ $x \otimes y \leq x \otimes z$ *iff (if and only if)* $y \leq z$,
$(P8)$ $x \otimes 1 = x$.

A binary operation $\oplus$ is said to be the triangular conorm (t-conorm for short) if $\oplus$ satisfies (P5), (P6), (P7) and
$(P9)$ $x \oplus 0 = 0$.

For a triangular norm $\otimes$, if $x \oplus y = N(N(x) \otimes N(y))$ $(x, y \in L)$, then the t-conorm $\oplus$ is called the dual t-conorm of $\otimes$. Similarly, if $x \otimes y = N(N(x) \oplus N(y))$ $(x, y \in L)$, then $\otimes$ is called the dual t-conorm of $\oplus$.

Definition 2.5. [13] *A t-norm $\otimes$ is left-continuous, if $\otimes$ satisfies for any $x_i, y \in L$ (where Q is a set of subscripts),*

$$(\bigvee_{i \in Q} x_i) \otimes y = \bigvee_{i \in Q} (x_i \otimes y). \tag{5}$$

The t-conorm $\oplus$ is right-continuous, if $\oplus$ meets for any $x_i, y \in L$,

$$(\bigwedge_{i \in Q} x_i) \oplus y = \bigwedge_{i \in Q} (x_i \oplus y). \tag{6}$$

Proposition 2.6. [13] *A t-conorm is left-continuous iff its dual t-conorm is right-continuous.*

Definition 2.7. [18] *A mapping $\to$ from L^2 to L is called an R-implication, if there exists a left-continuous t-norm $\otimes$ such that*

$$x \to y = \sup\{z \in [0,1] \mid x \otimes z \le y\}, \quad x, y \in L. \tag{7}$$

Proposition 2.8. [36] *If the t-conorm $\oplus$ is right-continuous, then there is a binary operation $\ominus$ (the $\oplus$-the coresiduum), so that $(\oplus, \ominus)$ constructs a co-adjoint pair, that is,*

$$x \le y \oplus z \quad \text{iff} \quad x \ominus z \le y, \tag{8}$$

where $\ominus$ is given by

$$x \ominus y = \wedge\{z \in L | x \le z \oplus y\}. \tag{9}$$

Definition 2.9. [36] *The operations $\to$, $\oplus$, $\ominus$ are referred to as associative operators of t-norm $\otimes$, whenever $(\otimes, \to)$ is an adjoint pair, and $(\oplus, \ominus)$ is a co-adjoint pair, and $\oplus$ is the dual t-conorm of $\otimes$.*

Proposition 2.10. [36] *If $\to$, $\oplus$, $\ominus$ are associative operators of the t-norm $\otimes$, then $x \ominus y = 1 - (1 - y) \to (1 - x)$.*

Example 2.11. *Here are four important t-norms. The first three are continuous, while the last one is left-continuous.*
(i) $a \otimes_G b = a \wedge b$ *(Gödel t-norm),*
(ii) $a \otimes_{Lu} b = (a + b - 1) \vee 0$ *(Lukasiewicz t-norm),*
(iii) $a \otimes_\pi = ab$ *(Product t-norm)*
(iv) $a \otimes_0 b = \begin{cases} 0, & a + b \le 1; \\ a \wedge b, & a + b > 1. \end{cases}$ *(Nilpotent minimum t-norm).*
The associate operators of these four t-norms are in turn as follows:
(i) $a \to_G b = \begin{cases} 1, & a \le b; \\ b, & a > b. \end{cases}$ $\quad a \oplus_G b = a \vee b.$ $\quad b \ominus_G a = \begin{cases} 0, & b \le a, \\ b, & b > a. \end{cases}$
(ii) $a \to_{Lu} b = (1 - a + b) \wedge 1.$ $\quad a \oplus_{Lu} b = (a + b) \wedge 1.$ $\quad b \ominus_{Lu} a = (b - a) \vee 0.$
(iii) $a \to_\pi b = \begin{cases} 1, & a \le b; \\ \frac{b}{a}, & a > b. \end{cases}$ $\quad a \oplus_\pi b = a + b - ab.$ $\quad b \ominus_\pi a = \begin{cases} 0, & b \le a; \\ \frac{b-a}{1-a}, & b > a. \end{cases}$
(iv) $a \to_0 b = \begin{cases} 1, & a \le b; \\ (1 - a) \vee b, & a > b. \end{cases}$ $\quad a \oplus_0 b = \begin{cases} 1, & a + b \ge 1; \\ a \vee b, & a + b < 1. \end{cases}$ $\quad b \ominus_0 a = \begin{cases} 0, & b \le a; \\ b \wedge (1 - a), & b > a. \end{cases}$

Lemma 2.12. [11, 32] *Let $\to$ be an R-implication obtained from a left-continuous t-norm $\otimes$, then $\to$ satisfies (P1), (P2), (P3), (P4) as well as the following conditions:*
$(P10)$ $x \le y \Longleftrightarrow x \to y = 1$,
$(P11)$ $1 \to x = x$,
$(P12)$ $\to$ *is left-continuous w.r.t. the first variable and right-continuous w.r.t. the second variable,*
$(P13)$ $x \to (y \to z) = y \to (x \to z)$,
$(P14)$ $(x \otimes y) \to z = x \to (y \to z)$,
$(P15)$ $x \le y \to z \Longleftrightarrow y \le x \to z$,
$(P16)$ $(\sup_{x \in X} x) \to y = \inf_{x \in X}(x \to y)$,
$(P17)$ $y \to (\inf_{x \in X} x) = \inf_{x \in X}(y \to x)$,
in which $x, y, z \in [0,1]$ and $X \subset [0,1]$, $X \ne \varnothing$.

Definition 2.13. [3] *An intuitionistic fuzzy set A over the non-empty domain X is characterized by $A = \{\langle x, \pi_A(x), \tau_A(x) \mid x\ inX\rangle\}$, in which $\pi_A:\ X \to [0,1]$, $\tau_A : X \to [0,1]$ and $0 \le \pi_A(x) + \tau_A(x) \le 1$, $x \in X$. Here $\pi_A(x)$ and $\tau_A(x)$ represent a membership function and a non-membership function from X to A in turn. Obviously the intuitionistic fuzzy set A on X is rewritten as $A(x) = (a, b)$, $0 \le a + b \le 1$, $a, b \in [0,1]$, $x \in X$.*

As an extension of fuzzy sets, intuitionistic fuzzy sets extend the domain from [0,1] to the triangular domain $L^* = \{(a, b) \in [0,1]^2 | a + b \le 1\}$.

We denote $IF(X)$ as the set of all of the intuitions fuzzy sets on X. When $\pi_A(x) = 1 - \tau_A(x)$, the intuitional fuzzy set becomes a fuzzy set.

Definition 2.14. [36] *For $a, b \in L^*$ where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, we define the partial order relation on L^* as $a \le b \iff a_1 \le b_1, a_2 \ge b_2$. Obviously, $a \wedge b = (a_1 \wedge b_1, a_2 \vee b_2)$, $a \vee b = (a_1 \vee b_1, a_2 \wedge b_2)$.*

$0^* = (0,1)$ and $1^* = (1,0)$ are the smallest and largest elements on L^*, in turn. It is easy to show that $(L^*, \leq)$ is a complete lattice.

Definition 2.15. [36] *For $a, b \in L^*$ where $a = (a_1, a_2)$ and $b = (b_1, b_2)$, the binary operator $\otimes_{L^*}$ and $\oplus_{L^*}$ are characterized as follows:*

$$a \otimes_{L^*} b = (a_1 \otimes b_1, a_2 \oplus b_2), \tag{10}$$

$$a \oplus_{L^*} b = (a_1 \oplus b_1, a_2 \otimes b_2), \tag{11}$$

in which $\oplus$ is the dual t-conorm of the t-norm $\otimes$. Then $\otimes_{L^*}$ is said to be an intuitionistic t-norm derived from t-norm $\otimes$, while $\oplus_{L^*}$ is known as an intuitionistic t-conorm derived from t-norm $\oplus$.

Proposition 2.16. [36] *$(L^*, \otimes_{L^*}, 1^*)$ is a commutative monoid, and $\otimes_{L^*}$ is isotone; $(L^*, \oplus_{L^*}, 0^*)$ is a commutative monoid, and $\oplus_{L^*}$ is isotone.*

Proposition 2.17. [36] *If the t-norm $\otimes$ is left-continuous, then*
(i) $\otimes_{L^*}$ is a left-continuous intuitionistic t-norm on L^*, that is (where I is a set of subscripts), $(\vee_{i\in I} a_i) \otimes_{L^*} c = \vee_{i\in I}(a_i \otimes_{L^*} c)$, $a_i, c \in L^*$;
(ii) $\oplus_{L^*}$ is a right-continuous intuitionistic t-conorm on L^*, that is (where I is a set of subscripts), $(\wedge_{i\in I} a_i) \oplus_{L^*} c = \wedge_{i\in I}(a_i \oplus_{L^*} c)$, $a_i, c \in L^*$.

Proposition 2.18. [10] *Suppose that $\otimes_{L^*}$ is an intuitionistic t-norm generated by a left-continuous t-norm $\otimes$, then there is a binary operator $\to_{L^*}$ to make*

$$a \otimes_{L^*} b \leq c \iff a \leq b \to_{L^*} c, \tag{12}$$

in which $\to_{L^*}$ is computed by

$$a \to_{L^*} b = \vee\{d \in L^* | d \otimes_{L^*} a \leq b\}. \tag{13}$$

Definition 2.19. [36] *If $(\otimes_{L^*}, \to_{L^*})$ meets (12), then $(\otimes_{L^*}, \to_{L^*})$ is called an intuitionistic adjoint pair, and $\to_{L^*}$ is called a residual intuitionistic implication.*

Proposition 2.20. [36] *Suppose that $\otimes_{L^*}$ is an intuitionistic t-norm and $(\otimes_{L^*}, \to_{L^*})$ is an intuitionistic adjoint pair, then $(a, b, c, a_i, b_i \in L^*)$*
$(P18)$ $a \to_{L^*} b = 1^* \iff a \leq b$,
$(P19)$ $c \leq a \to_{L^*} b \iff a \leq c \to_{L^*} b$,
$(P20)$ $c \to_{L^*} (a \to_{L^*} b) = a \to_{L^*} (c \to_{L^*} b)$,
$(P21)$ $1^* \to_{L^*} a = a$,
$(P22)$ $b \to_{L^*} (\wedge_{i\in I} a_i) = \wedge_{i\in I}(b \to_{L^*} a_i)$,
$(P23)$ $\vee_{i\in I} b_i \to_{L^*} a = \vee_{i\in I}(b_i \to_{L^*} a)$,
$(P24)$ $\to_{L^*}$ *decreases w.r.t. the first variable and increases w.r.t. the second variable.*

Proposition 2.21. [36] *If $\to_{L^*}$ is a residual intuitionistic implication generated by a left-continuous t-norm $\otimes$, then $(a, b \in L^*,\ a = (a_1, a_2),\ b = (b_1, b_2),)$*

$$a \to_{L^*} b = ((a_1 \to b_1) \wedge (1 - (b_2 \ominus a_2)),\ b_2 \ominus a_2). \tag{14}$$

Proposition 2.22. [36] *If $\to_{L^*}$ is a residual intuitionistic implication reduced by a left-continuous t-norm $\otimes$, then $a \leq (a \to_{L^*} 0^*) \to_{L^*} 0^*$ $(a \in L^*)$.*

Definition 2.23. [22] *E is an entropy measure of $IF(U)$ if it meets the following axiomatic requirements:*
(i) $E(A) = 0$ *iff* A *is a crisp set;*
(ii) $E(A) = 1$ *iff* $\pi_A(u) = \tau_A(u)$ *for any* $u \in U$;
(iii) $E(A) = E(A^c)$, *where A^c is the complement of A (i.e.,$A^c = \{< u, \tau_A(u), \pi_A(u) > | u \in U\}$);*
(iv) $E(A) \leq E(B)$, *if for any $u \in U$,*

$$\pi_A(u) \geq \pi_B(u),\ \tau_B(u) \geq \tau_A(u)\ for\ \pi_B(u) \geq \tau_B(u),$$

or

$$\pi_A(u) \leq \pi_B(u),\ \tau_B(u) \leq \tau_A(u)\ for\ \pi_B(u) \leq \tau_B(u).$$

There are many examples of fuzzy entropies. For example, the following is a commonly encountered fuzzy entropy:

$$E_{\mathrm{ZL}}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n} |\pi_A(u_i) - \tau_A(u_i)|\,. \tag{15}$$

3 The IFESI algorithm for IFMP

The IFMP problem is described as follows:

Assuming	$A(x) \to_* B(y)$	$\cdots$ Major premise
Given	$A^*(x)$	$\cdots$ Minor premise
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Calculate	$B^*(y)$	$\cdots$ Conclusion

Here A and A^* are two intuitionistic fuzzy sets on X. B and B^* are two intuitionistic fuzzy sets on Y. $\to_*$ is a residual intuitionistic fuzzy implication on L^*. Besides, $A(x) = (\pi_A(x), \tau_A(x)),\ B(y) = (\pi_B(y), \tau_B(y)),\ A^*(x) = (\pi_A^*(x), \tau_A^*(x))$.

From the idea of the IFESI algorithm for IFMP, we establish the following principle:

Basic IFESI principle for IFMP: The result B^* of the IFMP problem is the intuitionistic fuzzy set with maximum entropy such that

$$(A(x) \to_{*1} B(y)) \to_{*2} (A^*(x) \to_{*1} B^*(y)), \tag{16}$$

gets its maximum for any $x \in X, y \in Y$, in which $\to_{*1}, \to_{*2}$ are two residual intuitionistic fuzzy implications.

Definition 3.1. *Let $A, A^* \in IF(X)$, $B \in IF(Y)$, if B^* (in $IF(Y)$) makes (16) take its maximum for any $x \in X, y \in Y$. Then B^* is said to be an IFESI solution for IFMP.*

Definition 3.2. *Assume that $A, A^* \in IF(X)$, $B \in IF(Y)$, and that nonempty set $\mathbb{B}$ is the set of all IFESI solutions for IFMP, and finally that D^* is the intuitionistic fuzzy set with maximum entropy in $\mathbb{B}$. Then D^* is said to be a formal IFESI solution for IFMP.*

Theorem 3.3. *Assuming that $\to_{*1}, \to_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, then the formal IFESI solution for IFMP is as follows ($y \in Y$):*

$$B^*(y) = \bigvee_{x\in X}\{A^*(x) \otimes_{*1} (A(x) \to_{*1} B(y))\} \vee (0.5, 0.5). \tag{17}$$

Proof. Denote $D^*(y) = \bigvee_{x\in X}\{A^*(x) \otimes_{*1} (A(x) \to_{*1} B(y))\}$ for $y \in Y$. Let us first prove that D^* is the intuitionistic fuzzy set that maximizes (16). Obviously the maximum of (16) is 1^*.

Note that $(\otimes_{*1}, \to_{*1})$ is an intuitionistic adjoint pair. According to the expression of D^*, Proposition 2.18, and Proposition 2.20, it is known that ($x \in X,\ y \in Y$)

$$A^*(x) \otimes_{*1} (A(x) \to_{*1} B(y)) \le D^*(y),\ (A(x) \to_{*1} B(y)) \otimes_{*1} A^*(x) \le D^*(y),$$
$$A(x) \to_{*1} B(y) \le A^*(x) \to_{*1} D^*(y),\ (A(x) \to_{*1} B(y)) \to_{*2} (A^*(x) \to_{*1} D^*(y)) = 1^*.$$

So D^* is an IFESI solution.

Furthermore, we prove that D^* is the minimum of all IFESI solutions for IFMP.

Suppose that C is any IFESI solution for IFMP. Notice that $(\otimes_{*1}, \to_{*1})$ is an intuitionistic adjoint pair. Then from Proposition 2.18 and Proposition 2.20, one has

$$(A(x) \to_{*1} B(y)) \to_{*2} (A^*(x) \to_{*1} C(y)) = 1^*,\ A(x) \to_{*1} B(y) \le A^*(x) \to_{*1} C(y),$$
$$(A(x) \to_{*1} B(y)) \otimes_{*1} A^*(x) \le C(y),\ A^*(x) \otimes_{*1} (A(x) \to_{*1} B(y)) \le C(y).$$

It can be known that $C(y)$ is an upper bound of $\{A^*(x) \otimes_{*1} (A(x) \to_{*1} B(y)) \mid x \in X\},\ y \in Y$.

So $D^*(y) \le C(y)$ ($y \in Y$) and D^* is the minimum of $\mathbb{B}$.

Because $D^*(y) \le B^*(y)$ ($y \in Y$), we have $B^* \in \mathbb{B}$.

Finally, we further prove that B^* is the solution with maximum entropy in $\mathbb{B}$. Suppose that B_k is any intuitionistic fuzzy set in $\mathbb{B}$. Obviously $D^*(y) \le B_k(y)$ ($y \in Y$).

It can be split into the following three scenarios ($y \in Y$).

(i) $D^*(y) \le B_k(y) \le (0.5, 0.5)$. Note that $B^*(y) = D^*(y) \vee (0.5, 0.5)$. At this time we have $B^*(y) = (0.5, 0.5)$. Then we can view it as $\pi_{B^*} \le \tau_{B^*}$. Hence we have $\pi_{B_k} \le \pi_{B^*},\ \tau_{B^*} \le \tau_{B_k},\ for\ \pi_{B^*} \le \tau_{B^*}$.

(ii) $D^*(y) \le (0.5, 0.5) \le B_k(y)$. Here we also have $B^*(y) = (0.5, 0.5)$. Then we can view it as $\pi_{B^*} \ge \tau_{B^*}$. So we get $\pi_{B_k} \ge \pi_{B^*},\ \tau_{B^*} \ge \tau_{B_k},\ for\ \pi_{B^*} \ge \tau_{B^*}$.

(iii) $(0.5, 0.5) \le D^*(y) \le B_k(y)$. Then one has $B^* = D^*$, so we can get $0.5 \le \pi_{B^*} \le \pi_{B_k},\ 0.5 \ge \tau_{B^*} \ge \tau_{B_k}$, then $\pi_{B^*} \ge \tau_{B^*}$. So we have $\pi_{B_k} \ge \pi_{B^*},\ \tau_{B^*} \ge \tau_{B_k},\ for\ \pi_{B^*} \ge \tau_{B^*}$.

For these three scenarios, it follows from Definition 2.12 that one has $E(B_k) \le E(B^*)$.

To sum up, B^* is the solution with maximum entropy in $\mathbb{B}$, i.e., the formal IFESI solution for IFMP.

□

Proposition 3.4. *Assuming that $\rightarrow_{*1}, \rightarrow_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, and that $\otimes_{*1}$ corresponds to $\otimes_1$, and that $\rightarrow_1$, $\oplus_1$, $\ominus_1$ are associative operators of $\otimes_1$, then the formal IFESI solution $B^*(y) = (\pi_{B^*}(y), \tau_{B^*}(y))$ for IFMP can be expressed as ($y \in Y$):*

$$\begin{aligned} \pi_{B^*}(y) &= \vee_{x\in X}\{\pi_{A^*}(x) \otimes_1 ((\pi_A(x) \rightarrow_1 \pi_B(y)) \wedge (1 - (\tau_B(y) \ominus_1 \tau_A(x))))\} \vee 0.5, \\ \tau_{B^*}(y) &= \wedge_{x\in X}\{\tau_{A^*}(x) \oplus_1 (\tau_B(y) \ominus_1 \tau_A(x))\} \wedge 0.5. \end{aligned} \tag{18}$$

Proof. From Proposition 2.21, we have $A(x) \rightarrow_{*1} B(y) = ((\pi_A(x) \rightarrow_1 \pi_B(y)) \wedge (1 - (\tau_B(y) \ominus_1 \tau_A(x))),\ \tau_B(y) \ominus_1 \tau_A(x))$.

From Definition 2.9 and Definition 2.10, we know that $a \vee b = (a_1 \vee b_1, a_2 \wedge b_2)$ and $a \otimes_{L^*} b = (a_1 \otimes b_1, a_2 \oplus b_2)$ $(a, b \in L^*)$.

From Theorem 3.3, the formal IFESI solution for IFMP is $B^*(y) = \bigvee_{x\in X}\{A^*(x) \otimes_{*1} (A(x) \rightarrow_{*1} B(y))\} \vee (0.5, 0.5)$ $(y \in Y)$.

Together we get that (18) holds. □

Definition 3.5. *Aiming at an algorithm to deal with IFMP problem, when the condition (C) is met, if $A^* = A$ means $B^* = B$, then the algorithm is said to be C-reductive.*

Theorem 3.6. *Assuming that $\rightarrow_{*1}, \rightarrow_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, when the following two conditions are satisfied:*

($P25$) $E_y = \{x \in X \mid (0.5, 0.5) \leq A(x) \otimes_{*1} (A(x) \rightarrow_{*1} B(y))\} \neq \varnothing$ holds for any $y \in Y$,

($P26$) there exists $x_0 \in E_y$ to let $A(x_0) = 1^*$,

then the IFESI algorithm for IFMP is C-reductive, in which

$$C \text{ means } (P25) + (P26).$$

Proof. Suppose that $A^* = A$. It follows from Theorem 3.3 that the formal IFESI solution for IFMP is

$$B^*(y) = \bigvee_{x\in X}\{A^*(x) \otimes_{*1} (A(x) \rightarrow_{*1} B(y))\} \vee (0.5, 0.5) = \bigvee_{x\in E_y}\{A(x) \otimes_{*1} (A(x) \rightarrow_{*1} B(y))\}. \tag{19}$$

On the one hand, we prove $B(y) \leq B^*(y)$ $(y \in Y)$.

Note that $(\otimes_{*1}, \rightarrow_{*1})$ is an intuitionistic adjoint pair and that there exists $x_0 \in E_y$ such that $A(x_0) = 1^*$. According to the expression of (19), Proposition 2.18, and Proposition 2.20, one has $(y \in Y)$

$$\begin{aligned} &A(x_0) \otimes_{*1} (A(x_0) \rightarrow_{*1} B(y)) \leq B^*(y),\ (A(x_0) \rightarrow_{*1} B(y)) \otimes_{*1} A(x_0) \leq B^*(y), \\ &A(x_0) \rightarrow_{*1} B(y) \leq A(x_0) \rightarrow_{*1} B^*(y),\ 1^* \rightarrow_{*1} B(y) \leq 1^* \rightarrow_{*1} B^*(y),\ B(y) \leq B^*(y). \end{aligned}$$

On the other hand, we validate $B^*(y) \leq B(y)$ $(y \in Y)$.

From another viewpoint, $A(x) \rightarrow_{*1} B(y) \leq A(x) \rightarrow_{*1} B(y)$ obviously holds for any $x \in X, y \in Y$. Because $(\otimes_{*1}, \rightarrow_{*1})$ is an intuitionistic adjoint pair, we have $(A(x) \rightarrow_{*1} B(y)) \otimes_{*1} A(x) \leq B(y)$ $(x \in X, y \in Y)$. That is $A(x) \otimes_{*1} (A(x) \rightarrow_{*1} B(y)) \leq B(y)$. Then we get from (19) that $B^*(y) \leq B(y)$ $(y \in Y)$.

Together we have $B^*(y) = B(y)$ $(y \in Y)$. As a result, the IFESI algorithm for IFMP is C-reductive. □

4 The IFESI algorithm for IFMT

The IFMT problem is described as follows:

Assuming	$A(x) \rightarrow_* B(y)$	⋯ Major premise
Given	$B^*(y)$	⋯ Minor premise
− − − − − −	− − − − − − − − − −	− − − − − − − − − − −
Calculate	$A^*(x)$	⋯ Conclusion

Here A and A^* are the intuitionistic fuzzy sets on X. B and B^* are the intuitionistic fuzzy sets on Y. $\rightarrow_*$ is the residual intuitionistic fuzzy implication on L^*. Besides, $A(x) = (\pi_A(x), \tau_A(x))$, $B(y) = (\pi_B(y), \tau_B(y))$, $B^*(y) = (\pi^*_B(y), \tau^*_B(y))$.

From the idea of the IFESI algorithm for IFMT, we formulate the following principle:

Basic IFESI principle for IFMT: The result A^* of the IFMT problem is the intuitionistic fuzzy set with maximum entropy such that (16)

gets its maximum for any $x \in X, y \in Y$, in which $\rightarrow_{*1}, \rightarrow_{*2}$ are two residual intuitionistic fuzzy implications.

Definition 4.1. *Let $A \in IF(X)$, $B, B^* \in IF(Y)$, if A^* (in $IF(X)$) makes (16) take its maximum for any $x \in X, y \in Y$. Then A^* is said to be an IFESI solution for IFMT.*

Definition 4.2. *Assume that $A \in IF(X)$, $B, B^* \in IF(Y)$, and that nonempty set $\mathbb{A}$ is the set of all IFESI solutions for IFMT, and finally that C^* is the intuitionistic fuzzy set with maximum entropy in $\mathbb{A}$. Then C^* is said to be a formal IFESI solution for IFMT.*

Theorem 4.3. *Assuming that $\rightarrow_{*1}, \rightarrow_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, then the formal IFESI solution for IFMT is as follows ($x \in X$):*

$$A^*(x) = \bigwedge_{y\in Y}\{(A(x) \rightarrow_{*1} B(y)) \rightarrow_{*1} B^*(y)\} \wedge (0.5, 0.5). \tag{20}$$

Proof. Denote $C^*(x) = \bigwedge_{y\in Y}\{(A(x) \rightarrow_{*1} B(y)) \rightarrow_{*1} A^*(x)\}$ ($x \in X$).

Let us first prove that A^* is the intuitionistic fuzzy set that maximizes (16). Note that the maximum of (16) is 1^*. Note that $(\otimes_{*1}, \rightarrow_{*1})$ is an intuitionistic adjoint pair. According to the expression of C^*, Proposition 2.18, and Proposition 2.20, one has ($x \in X,\ y \in Y$)

$$\begin{aligned}
&C^*(x) \leq (A(x) \rightarrow_{*} B(y)) \rightarrow_{*1} B^*(y),\\
&A(x) \rightarrow_{*1} B(y) \leq C^*(x) \rightarrow_{*_1} B^*(y),\\
&(A(x) \rightarrow_{*_1} B(y)) \rightarrow_{*_2} (C^*(x) \rightarrow_{*_1} B^*(y)) = 1^*.
\end{aligned}$$

Furthermore, we validate that C^* is the maximum of all IFESI solutions for IFMT.

Suppose that C is any IFESI solution for IFMT. Notice that $(\otimes_{*1}, \rightarrow_{*1})$ is an intuitionistic adjoint pair. Then from Proposition 2.18 and Proposition 2.20, one has

$$\begin{aligned}
&(A(x) \rightarrow_{*_1} B(y)) \rightarrow_{*_2} (C(x) \rightarrow_{*_1} B^*(y)) = 1^*,\\
&A(x) \rightarrow_{*_1} B(y) \leq C(x) \rightarrow_{*_1} B^*(y),\\
&C(x) \leq (A(x) \rightarrow_{*_1} B(y)) \rightarrow_{*_1} B^*(y).
\end{aligned}$$

We can find that $C(x)$ is a lower bound of $\{(A(x) \rightarrow_{*_1} B(y)) \rightarrow_{*_1} B^*(y) \mid y \in Y\},\ x \in X$.

Then $C(x) \leq C^*(x)$ ($x \in X$) and C^* is the maximum of $\mathbb{A}$.

Since $A^*(x) \leq C^*(x)$ ($x \in X$), then $A^*(x) \in \mathbb{A}$.

Finally, we further prove that A^* is the solution with maximum entropy in $\mathbb{A}$. Suppose that A_k is any intuitionistic fuzzy set in $\mathbb{A}$. Obviously $A_k(x) \leq C^*(x)$ ($x \in X$).

It can be split into the following three scenarios ($x \in X$).

(i) $(0.5, 0.5) \leq A_k(x) \leq C^*(x)$. Note that $A^*(x) = C^*(x) \wedge (0.5, 0.5)$. Here we have $A^*(x) = (0.5, 0.5)$. Then we can view it as $\pi_{A^*} \geq \tau_{A^*}$. Hence we have $\pi_{A_k} \geq \pi_{A^*},\ \tau_{A^*} \geq \tau_{A_k},\ for\ \pi_{A^*} \geq \tau_{A^*}$.

(ii) $A_k(x) \leq (0.5, 0.5) \leq C^*(x)$. Here we also have $A^*(x) = (0.5, 0.5)$. Then we can view it as $\pi_{A^*} \leq \tau_{A^*}$. So we get $\pi_{A_k} \leq \pi_{A^*},\ \tau_{A^*} \leq \tau_{A_k},\ for\ \pi_{A^*} \leq \tau_{A^*}$.

(iii) $A_k(x) \leq C^*(x) \leq (0.5, 0.5)$. Then one has $A^* = C^*$, so we can get $\pi_{A_k} \leq \pi_{A^*} \leq 0.5,\ \tau_{A_k} \geq \tau_{A^*} \geq 0.5$, then $\pi_{A^*} \leq \tau_{A^*}$. So we have $\pi_{A_k} \leq \pi_{A^*},\ \tau_{A^*} \leq \tau_{A_k},\ for\ \pi_{A^*} \leq \tau_{A^*}$.

For these three scenarios, it follows from Definition 2.12 that one has $E(A_k) \leq E(A^*)$.

To sum up, A^* is the solution with maximum entropy in $\mathbb{A}$, i.e., the formal IFESI solution for IFMT. □

Proposition 4.4. *Assuming that $\rightarrow_{*1}, \rightarrow_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, and that $\otimes_{*1}$ corresponds to $\otimes_1$, and that $\rightarrow_1$, $\oplus_1$, $\ominus_1$ are associative operators of $\otimes_1$, then the formal IFESI solution $A^*(x) = (\pi_{A^*}(x), \tau_{A^*}(x))$ for IFMT can be expressed as ($x \in X$):*

$$\begin{aligned}
&\pi_{A^*}(x) = \wedge_{y\in Y}\{[((\pi_A(x) \rightarrow_1 \pi_B(y)) \wedge (1 - (\tau_B(y) \ominus_1 \tau_A(x)))) \rightarrow_1 \pi_{B^*}(y)] \wedge [1 - (\tau_{B^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x)))]\} \wedge 0.5,\\
&\tau_{A^*}(x) = \vee_{y\in Y}\{\tau_{B^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x))\} \vee 0.5.
\end{aligned} \tag{21}$$

Proof. It follows from Proposition 2.21 that $A(x) \rightarrow_{*1} B(y) = ((\pi_A(x) \rightarrow_1 \pi_B(y)) \wedge (1 - (\tau_B(y) \ominus_1 \tau_A(x))),\ \tau_B(y) \ominus_1 \tau_A(x))$.

From Definition 2.14, Definition 2.15, we know $a \vee b = (a_1 \vee b_1, a_2 \wedge b_2)$, and $a \otimes_{L^*} b = (a_1 \otimes b_1, a_2 \oplus b_2)$ ($a, b \in L^*$).

From Theorem 4.3, the formal IFESI solution for IFMT is $A^*(x) = \bigwedge_{y\in Y}\{(A(x) \rightarrow_{*1} B(y)) \rightarrow_{*1} B^*(y)\} \wedge (0.5, 0.5)$ ($x \in X$).

Together we get that (21) holds. □

Definition 4.5. *Aiming at an algorithm to deal with IFMT problem, when the condition (C) is met, if $B^* = B$ implies $A^* = A$, then the algorithm is said to be C-reductive.*

Theorem 4.6. *Assuming that $\rightarrow_{*1}, \rightarrow_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, when the following two conditions are satisfied:*

$(P27)$ $E_x = \{y \in Y \mid (0.5, 0.5) \geq (A(x) \rightarrow_{*1} B(y)) \rightarrow_{*1} B(y)\} \neq \varnothing$ holds for any $x \in X$,

$(P28)$ there exists $y_0 \in E_x$ to let $B(y_0) = 0^*$,

$(P29)$ $a \rightarrow_{*1} 0^* = 1 - a$ holds for any $a \in L^*$,

then the IFESI algorithm for IFMT is C-reductive, in which

$$C \text{ means } (P27) + (P28) + (P29).$$

Proof. Suppose that $B^* = B$. It follows from Theorem 4.3 that the formal IFESI solution for IFMT is

$$A^*(x) = \bigwedge_{y \in Y} \{(A(x) \rightarrow_{*1} B(y)) \rightarrow_{*1} B^*(y)\} \wedge (0.5, 0.5) = \bigwedge_{y \in E_x} \{(A(x) \rightarrow_{*1} B(y)) \rightarrow_{*1} B(y)\}. \tag{22}$$

From one point of view, we prove $A(x) \geq A^*(x)$ $(x \in X)$.

Note that $(\otimes_{*1}, \rightarrow_{*1})$ is an intuitionistic adjoint pair and that there exists $y_0 \in E_x$ such that $B(y_0) = 0^*$.

Then we get from the expression of (22), Proposition 2.18 and Proposition 2.20 that $(x \in X)$

$$\begin{aligned}
&(A(x) \rightarrow_{*1} B(y_0)) \rightarrow_{*1} B(y_0) \geq A^*(x), \\
&(A(x) \rightarrow_{*1} 0^*) \rightarrow_{*1} 0^* \geq A^*(x), \\
&A(x) \geq A^*(x).
\end{aligned}$$

From another point of view, we validate $A(x) \leq A^*(x)$ $(x \in X)$.

Notice that $A(x) \rightarrow_{*1} B(y) \leq A(x) \rightarrow_{*1} B(y)$ obviously holds for any $x \in X, y \in Y$.

From Proposition 2.20, we have $A(x) \leq (A(x) \rightarrow_{*1} B(y)) \rightarrow_{*1} B(y)$ $(x \in X, y \in Y)$.

Then we get from (22) that $A(x) \leq A^*(x)$ $(x \in X)$.

Together we have $A^*(x) = A(x)$ $(x \in X)$. As a result, the IFESI algorithm for IFMT is C-reductive. □

5 Continuity of the IFESI algorithm

In this section we assume that the universes X, Y are finite sets, i.e., $X = \{x_1, x_2, \cdots, x_m\}$, $Y = \{y_1, y_2, \cdots, y_n\}$.

Based on the distance of fuzzy sets, Atanassov introduced the distance metric NHD [4] of intuitionistic fuzzy set as follows:

$$NHD(A, A^*) = \frac{1}{2m} \sum_{i=1}^{m} [|\pi_A(x_i) - \pi_{A^*}(x_i)| + |\tau_A(x_i) - \tau_{A^*}(x_i)|].$$

Definition 5.1. *Assuming that X, Y are two nonempty finite fields, $A^*, A \in IF(X)$, $A \rightarrow_* B$ is the given rule, then the IFESI algorithm of IFMP problem can be regarded as a mapping h from $IF(X)$ to $IF(Y)$.*

(i) For any $\varepsilon > 0$, if there exists $\delta > 0$ such that $d(h(A_1^*), h(A_2^*)) < \varepsilon$ whenever $d(A_1^*, A_2^*) < \delta$ for any $A_1^*, A_2^* \in IF(X)$, then the algorithm is uniformly continuous in metric d.

(ii) For any $\varepsilon > 0$, if there exists $\delta > 0$ such that $d(h(A_1^*), h(A^*)) < \varepsilon$ whenever $d(A_1^*, A^*) < \delta$ for any $A_1^* \in IF(X)$, then the algorithm is continuous in metric d.

Definition 5.2. *Assuming that X, Y are two nonempty finite fields, $B^*, B \in IF(Y)$, $A \rightarrow_* B$ is the given rule , then the IFESI algorithm of IFMT problem can be regarded as a mapping h from $IF(Y)$ to $IF(X)$.*

(i) For any $\varepsilon > 0$, if there exists $\delta > 0$ such that $d(h(B_1^*), h(B_2^*)) < \varepsilon$ whenever $d(B_1^*, B_2^*) < \delta$ for any $B_1^*, B_2^* \in IF(Y)$, then the algorithm is uniformly continuous in metric d.

(ii) For any $\varepsilon > 0$, if there exists $\delta > 0$ such that $d(h(B_1^*), h(B^*)) < \varepsilon$ whenever $d(B_1^*, B^*) < \delta$ for any $B_1^* \in IF(Y)$, then the algorithm is uniformly continuous in metric d.

It is easy to prove Lemma 5.3 and Lemma 5.4.

Lemma 5.3. *$|a \wedge c - b \wedge c| \leq |a - b|$, $|a \vee c - b \vee c| \leq |a - b|$, where $a, b, c \in [0, 1]$.*

Lemma 5.4. *$|a \wedge c - b \wedge d| \leq |a - b| + |c - d|$, where $a, b, c, d \in [0, 1]$.*

Lemma 5.5. [12] *If the functions $f, g: U \to R$ are bounded, U is a non-empty set, and R is a set of real numbers, then for any $u \in U$, one has (i) $|\bigvee_{u\in U} f(u) - \bigvee_{u\in U} g(u)| \le \bigvee_{u\in U} |f(u)-g(u)|$; (ii) $|\bigwedge_{u\in U} f(u) - \bigwedge_{u\in U} g(u)| \le \bigvee_{u\in U} |f(u)-g(u)|$.*

Theorem 5.6. *For the same condition of Theorem 3.3, $(\otimes_{*1}, \to_{*1})$ is the intuitionistic adjoint pair generated by an t-norm $\otimes_1$. If the t-norm $\otimes_1$ is continuous, then the IFESI algorithm of IFMP is uniformly continuous in NHD.*

Proof. Note that the t-norm $\otimes_1$ is continuous, it follows that $\otimes_1$ is uniformly continuous w.r.t. its first variable in [0,1]. As a result, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, such that $|(\pi_{A_1^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x)))) - (\pi_{A_2^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x))))| < \varepsilon$ holds if $|\pi_{A_1^*}(x) - \pi_{A_2^*}(x)| < \delta_1$ $(x \in X,\ i = 1, 2, \cdots, n)$.

Since the t-norm $\otimes_1$ is continuous, it is easy to find that the corresponding t-conorm $\oplus_1$ is also continuous (noting that $x \oplus_1 y = 1 - (1-x) \otimes_1 (1-y)$ holds for any $x, y \in L$), and then $\oplus_1$ is uniformly continuous w.r.t. its first variable in [0,1]. For $\varepsilon > 0$, there exists $\delta_2 > 0$, such that $|(\tau_{A_1^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x))) - (\tau_{A_2^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x)))| < \varepsilon$ holds if $|\tau_{A_1^*}(x) - \tau_{A_2^*}(x)| < \delta_2$ $(x \in X,\ i = 1, 2, \cdots, n)$.

We take $\delta = \min\{\frac{\delta_1}{2m}, \frac{\delta_2}{2m}\}$.

Suppose that $NHD(A_1^*, A_2^*) < \delta$. Then we know from the formula of NHD that $|\pi_{A_1^*}(x) - \pi_{A_2^*}(x)| < \delta_1$ and that $|\tau_{A_1^*}(x) - \tau_{A_1^*}(x)| < \delta_2$ $(x \in X)$.

By virtue of Proposition 3.4, Lemma 5.3, Lemma 5.4, Lemma 5.5, we obtain

$$\begin{aligned}
NHD(B_1^*, B_2^*) =& \frac{1}{2n}\sum_{i=1}^{n}[|\pi_{B_1^*}(y_i) - \pi_{B_2^*}(y_i)| + |\tau_{B_1^*}(y_i) - \tau_{B_2^*}(y_i)|] \\
=& \frac{1}{2n}\sum_{i=1}^{n}[|\bigvee_{x\in X}(\pi_{A_1^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x)))) \vee 0.5 \\
& - \bigvee_{x\in X}(\pi_{A_2^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x)))) \vee 0.5|] \\
& + \frac{1}{2n}\sum_{i=1}^{n}[|\bigwedge_{x\in X}(\tau_{A_1^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x))) \wedge 0.5 - \bigwedge_{x\in X}(\tau_{A_2^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x))) \wedge 0.5|] \\
\le& \frac{1}{2n}\sum_{i=1}^{n}[|\bigvee_{x\in X}(\pi_{A_1^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x)))) \\
& - \bigvee_{x\in X}(\pi_{A_2^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x))))|] \\
& + \frac{1}{2n}\sum_{i=1}^{n}[|\bigwedge_{x\in X}(\tau_{A_1^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x))) - \bigwedge_{x\in X}(\tau_{A_2^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x)))|] \\
\le& \frac{1}{2n}\sum_{i=1}^{n}[\bigvee_{x\in X}|(\pi_{A_1^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x)))) \\
& - (\pi_{A_2^*}(x) \otimes_1 ((\pi_A(x) \to_1 \pi_B(y_i)) \wedge (1 - \tau_B(y_i) \ominus_1 \tau_A(x))))|] \\
& + \frac{1}{2n}\sum_{i=1}^{n}[\bigvee_{x\in X}|(\tau_{A_1^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x))) - (\tau_{A_2^*}(x) \oplus_1 (\tau_B(y_i) \ominus_1 \tau_A(x)))|] \\
<& \frac{1}{2n}\sum_{i=1}^{n}\bigvee_{x\in X}\varepsilon + \frac{1}{2n}\sum_{i=1}^{n}\bigvee_{x\in X}\varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

To sum up, there exists $\delta > 0$ such that $NHD(B_1^*, B_2^*) < \varepsilon$ if $NHD(A_1^*, A_2^*) < \delta$. Then the IFESI algorithm of IFMP is uniformly continuous in NHD. □

It is easy to find that if f is uniformly continuous then it is continuous, thus we obtain Theorem 5.7 from Theorem 5.6.

Theorem 5.7. *For the same condition of Theorem 3.3, $(\otimes_{*1}, \to_{*1})$ is the intuitionistic adjoint pair generated by a t-norm $\otimes_1$. If the t-norm $\otimes_1$ is continuous, then the IFESI algorithm of IFMP is continuous in NHD.*

Theorem 5.8. *For the same condition of Theorem 4.3, $(\otimes_{*1}, \to_{*1})$ is the intuitionistic adjoint pair generated by a t-norm $\otimes_1$, and $\to_1$, $\oplus_1$, $\ominus_1$ are associative operators of $\otimes_1$. If $\to_1$ is continuous w.r.t. the second variable, then the IFESI algorithm of IFMT is uniformly continuous in NHD.*

Proof. Note that $\to_1$ is continuous w.r.t. the second variable, then it follows that $\to_1$ is uniformly continuous w.r.t. its second variable in [0,1]. As a result, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, such that $|(((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_1^*}(y)) - (((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_2^*}(y))| < \frac{\varepsilon}{3}$ holds if $|\pi_{B_1^*}(y) - \pi_{B_2^*}(y)| < \delta_1$ $(y \in Y)$.

Since $\to_1$ is continuous w.r.t. the second variable, it is easy to find that the corresponding $\ominus_1$ is also continuous w.r.t. the first variable (noting that $x \ominus_1 y = 1 - (1 - y) \to_1 (1 - x)$ holds for any $x, y \in L$), and then $\ominus_1$ is uniformly continuous w.r.t. its first variable in [0,1]. For $\varepsilon > 0$, there exists $\delta_2 > 0$ such that $|(\tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))) - (\tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))| < \frac{\varepsilon}{3}$ holds if $|\tau_{B_1^*}(y) - \tau_{B_2^*}(y)| < \delta_2$ $(y \in Y)$.

We take $\delta = \min\{\frac{\delta_1}{2n}, \frac{\delta_2}{2n}\}$.

Suppose that $NHD(B_1^*, B_2^*) < \delta$. Then we know from the formula of NHD that $|\pi_{B_1^*}(y) - \pi_{B_2^*}(y)| < \delta_1$ and that $|\tau_{B_1^*}(y) - \tau_{B_1^*}(y)| < \delta_2$ $(y \in Y)$.

By virtue of Proposition 4.4, Lemma 5.3, Lemma 5.4, Lemma 5.5, we obtain

$$NHD(A_1^*, A_2^*) = \frac{1}{2m}\sum_{i=1}^{m}[|\pi_{A_1^*}(x_i) - \pi_{A_2^*}(x_i)| + |\tau_{A_1^*}(x_i) - \tau_{A_2^*}(x_i)|]$$

$$= \frac{1}{2m}\sum_{i=1}^{m}[|\bigwedge_{y\in Y}((((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_1^*}(y)) \wedge ((1 - \tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))))) \wedge 0.5$$
$$- \bigwedge_{y\in Y}((((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_2^*}(y)) \wedge ((1 - \tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))))) \wedge 0.5|]$$
$$+ \frac{1}{2m}\sum_{i=1}^{m}[|\bigvee_{y\in Y}(\tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))) \vee 0.5 - \bigvee_{y\in Y}(\tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))) \vee 0.5|]$$

$$\leq \frac{1}{2m}\sum_{i=1}^{m}[|\bigwedge_{y\in Y}((((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_1^*}(y)) \wedge ((1 - \tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))))$$
$$- \bigwedge_{y\in Y}((((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_2^*}(y)) \wedge ((1 - \tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))))|]$$
$$+ \frac{1}{2m}\sum_{i=1}^{m}[|\bigvee_{y\in Y}(\tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))) - \bigvee_{y\in Y}(\tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))|]$$

$$\leq \frac{1}{2m}\sum_{i=1}^{m}[\bigvee_{y\in Y}|(((((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_1^*}(y)) \wedge ((1 - \tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))))$$
$$- ((((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_2^*}(y)) \wedge ((1 - \tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))))|]$$
$$+ \frac{1}{2m}\sum_{i=1}^{m}[\bigvee_{y\in Y}|(\tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))) - (\tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))|]$$

$$\leq \frac{1}{2m}\sum_{i=1}^{m}[\bigvee_{y\in Y}|(((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_1^*}(y))$$
$$- (((\pi_A(x_i) \to_1 \pi_B(y)) \wedge (1 - \tau_B(y) \ominus_1 \tau_A(x_i))) \to_1 \pi_{B_2^*}(y))|]$$
$$+ \frac{1}{2m}\sum_{i=1}^{m}[\bigvee_{y\in Y}|(1 - \tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))) - (1 - \tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)))|] + \frac{1}{2m}\sum_{i=1}^{m}\bigvee_{y\in Y}\frac{\varepsilon}{3}$$

$$\leq \frac{1}{2m}\sum_{i=1}^{m}\bigvee_{y\in Y}\frac{\varepsilon}{3} + \frac{1}{2m}\sum_{i=1}^{m}[\bigvee_{y\in Y}|\tau_{B_1^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i)) - \tau_{B_2^*}(y) \ominus_1 (\tau_B(y) \ominus_1 \tau_A(x_i))|] + \frac{\varepsilon}{6}$$

$$\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2} < \varepsilon.$$

Summarizing above, there exists $\delta > 0$ such that $NHD(A_1^*, A_2^*) < \varepsilon$ if $NHD(B_1^*, B_2^*) < \delta$. Then the IFESI algorithm of IFMT is uniformly continuous in NHD. □

We similarly obtain Theorem 5.9 from Theorem 5.8.

Theorem 5.9. *For the same condition of Theorem 4.3, $(\otimes_{*1}, \to_{*1})$ is the intuitionistic adjoint pair generated by a t-norm $\otimes_1$, and $\to_1$, $\oplus_1$, $\ominus_1$ are associative operators of $\otimes_1$. If $\to_1$ is continuous w.r.t. the second variable, then the IFESI algorithm of IFMT is continuous in NHD.*

6 The α-IFESI algorithm of IFMP

α-IFESI principle for IFMP: The result B^* of the IFMP problem is the intuitionistic fuzzy set with maximum entropy such that

$$(A(x) \to_{*1} B(y)) \to_{*2} (A^*(x) \to_{*1} B^*(y)) \geq \alpha, \tag{23}$$

for any $x \in X, y \in Y$, in which $\to_{*1}, \to_{*2}$ are two residual intuitionistic fuzzy implications and $\alpha = (\alpha_1, \alpha_2) \in L^*$.

Definition 6.1. *Let $A, A^* \in IF(X)$, $B \in IF(Y)$, if B^* (in $IF(Y)$) makes (23) hold for any $x \in X, y \in Y$. Then B^* is said to be an α-IFESI solution for IFMP.*

Definition 6.2. *Assume that $A, A^* \in IF(X)$, $B \in IF(Y)$, and that nonempty set $\mathbb{B}_\alpha$ is the set of all α-IFESI solutions for IFMP, and finally that D^* is the intuitionistic fuzzy set with maximum entropy in $\mathbb{B}_\alpha$. Then D^* is said to be a formal α-IFESI solution for IFMP.*

Theorem 6.3. *Assuming that $\to_{*1}, \to_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, then the formal α-IFESI solution for IFMP is as follows ($y \in Y$):*

$$B^*(y) = \bigvee_{x \in X} \{A^*(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha)\} \vee (0.5, 0.5). \tag{24}$$

Proof. Denote $D^*(y) = \bigvee_{x \in X} \{A^*(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha)\}$ $(y \in Y)$.

Let us first validate that D^* is the intuitionistic fuzzy set that lets (23) hold.

Note that $(\otimes_{*1}, \to_{*1})$, $(\otimes_{*2}, \to_{*2})$ are two intuitionistic adjoint pairs. According to the expression of D^*, Proposition 2.18, and Proposition 2.20, it can be known that ($x \in X,\ y \in Y$)

$$\begin{aligned}
&A^*(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \leq D^*(y),\\
&((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \otimes_{*1} A^*(x) \leq D^*(y),\\
&(A(x) \to_{*1} B(y)) \otimes_{*2} \alpha \leq A^*(x) \to_{*1} D^*(y),\\
&\alpha \otimes_{*2} (A(x) \to_{*1} B(y)) \leq A^*(x) \to_{*1} D^*(y),\\
&\alpha \leq (A(x) \to_{*1} B(y)) \to_{*2} (A^*(x) \to_{*1} D^*(y)).
\end{aligned}$$

So D^* is an α-IFESI solution.

Furthermore, we prove that D^* is the minimum of all α-IFESI solutions for IFMP.

Suppose that C is any α-IFESI solution for IFMP. Notice that $(\otimes_{*1}, \to_{*1})$, $(\otimes_{*2}, \to_{*2})$ are two intuitionistic adjoint pairs. Then from Proposition 2.18 and Proposition 2.20, one has

$$\begin{aligned}
&\alpha \leq (A(x) \to_{*1} B(y)) \to_{*2} (A^*(x) \to_{*1} C(y)),\\
&\alpha \otimes_{*2} (A(x) \to_{*1} B(y)) \leq A^*(x) \to_{*1} C(y),\\
&(\alpha \otimes_{*2} (A(x) \to_{*1} B(y))) \otimes_{*1} A^*(x) \leq C(y),\\
&A^*(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \leq C(y).
\end{aligned}$$

It can be known that $C(y)$ is an upper bound of $\{A^*(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \mid x \in X\},\ y \in Y$.

So $D^*(y) \leq C(y)$ $(y \in Y)$ and D^* is the minimum of $\mathbb{B}_\alpha$.

Because $D^*(y) \leq B^*(y)$ $(y \in Y)$, we have $B^* \in \mathbb{B}_\alpha$.

Finally, we further prove that B^* is the solution with maximum entropy in $\mathbb{B}_\alpha$. Suppose that B_k is any intuitionistic fuzzy set in $\mathbb{B}_\alpha$. Obviously $D^*(y) \leq B_k(y)$ $(y \in Y)$.

It can be split into the following three scenarios $(y \in Y)$.

(i) $D^*(y) \leq B_k(y) \leq (0.5, 0.5)$. Note that $B^*(y) = D^*(y) \vee (0.5, 0.5)$. Then we have $B^*(y) = (0.5, 0.5)$. Then we can view it as $\pi_{B^*} \leq \tau_{B^*}$. Hence one has $\pi_{B_k} \leq \pi_{B^*},\ \tau_{B^*} \leq \tau_{B_k},\ for\ \pi_{B^*} \leq \tau_{B^*}$.

(ii) $D^*(y) \leq (0.5, 0.5) \leq B_k(y)$. Here we also have $B^*(y) = (0.5, 0.5)$. Then we can view it as $\pi_{B^*} \geq \tau_{B^*}$. So we get $\pi_{B_k} \geq \pi_{B^*},\ \tau_{B^*} \geq \tau_{B_k},\ for\ \pi_{B^*} \geq \tau_{B^*}$.

(iii) $(0.5, 0.5) \leq D^*(y) \leq B_k(y)$. Then one has $B^* = D^*$, so we can get $0.5 \leq \pi_{B^*} \leq \pi_{B_k},\ 0.5 \geq \tau_{B^*} \geq \tau_{B_k}$, then $\pi_{B^*} \geq \tau_{B^*}$. So we have $\pi_{B_k} \geq \pi_{B^*},\ \tau_{B^*} \geq \tau_{B_k},\ for\ \pi_{B^*} \geq \tau_{B^*}$.

For these three scenarios, it follows from Definition 2.12 that we get $E(B_k) \leq E(B^*)$.

Summarizing above, B^* is the solution with maximum entropy in $\mathbb{B}_\alpha$, i.e., the formal α-IFESI solution for IFMP. $\square$

Proposition 6.4. *Assuming that $\to_{*1}, \to_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, and that $\otimes_{*1}$ corresponds to $\otimes_1$, and that $\to_1$, $\oplus_1$, $\ominus_1$ are associative operators of $\otimes_1$, then the formal α-IFESI solution $B^*(y) = (\pi_{B^*}(y), \tau_{B^*}(y))$ for IFMP can be expressed as ($y \in Y$):*

$$\begin{aligned}\pi_{B^*}(y) &= \vee_{x\in X}\{\pi_{A^*}(x) \otimes_1 (((\pi_A(x) \to_1 \pi_B(y)) \wedge (1 - (\tau_B(y) \ominus_1 \tau_A(x)))) \otimes_{*2} \alpha_1)\} \vee 0.5,\\ \tau_{B^*}(y) &= \wedge_{x\in X}\{\tau_{A^*}(x) \oplus_1 ((\tau_B(y) \ominus_1 \tau_A(x)) \oplus_2 \alpha_2)\} \wedge 0.5.\end{aligned} \tag{25}$$

Proof. From Proposition 2.21, we have $A(x) \to_{*1} B(y) = ((\pi_A(x) \to_1 \pi_B(y)) \wedge (1 - (\tau_B(y) \ominus_1 \tau_A(x))),\ \tau_B(y) \ominus_1 \tau_A(x))$. From Definition 2.9 and Definition 2.10, we know that $a \vee b = (a_1 \vee b_1, a_2 \wedge b_2)$, $a \otimes_{*1} b = (a_1 \otimes_1 b_1, a_2 \oplus_1 b_2)$, $a \otimes_{*2} b = (a_1 \otimes_2 b_1, a_2 \oplus_2 b_2)$ $(a, b \in L^*)$. From Theorem 6.3, the formal α-IFESI solution for IFMP is $B^*(y) = \bigvee_{x\in X}\{A^*(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} (\alpha_1, \alpha_2))\} \vee (0.5, 0.5)$ $(y \in Y)$. Together we achieve that (25) holds. □

7 The α-IFESI algorithm of IFMT

α-IFESI principle for IFMT: The result A^* of the IFMT problem is the intuitionistic fuzzy set with maximum entropy such that (23) holds for any $x \in X, y \in Y$, in which $\to_{*1}, \to_{*2}$ are two residual intuitionistic fuzzy implications and $\alpha = (\alpha_1, \alpha_2) \in L^*$.

Definition 7.1. *Let $A \in IF(X)$, $B, B^* \in IF(Y)$, if A^* (in $IF(X)$) makes (23) hold for any $x \in X, y \in Y$. Then A^* is said to be an α-IFESI solution for IFMT.*

Definition 7.2. *Assume that $A \in IF(X)$, $B, B^* \in IF(Y)$, and that nonempty set $\mathbb{A}_\alpha$ is the set of all α-IFESI solutions for IFMT, and finally that C^* is the intuitionistic fuzzy set with maximum entropy in $\mathbb{A}_\alpha$. Then C^* is referred to as a formal α-IFESI solution for IFMT.*

Theorem 7.3. *Assuming that $\to_{*1}, \to_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, then the formal α-IFESI solution for IFMT is as follows ($x \in X$):*

$$A^*(x) = \bigwedge_{y\in Y}\{((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y)\} \wedge (0.5, 0.5). \tag{26}$$

Proof. Denote $C^*(x) = \bigwedge_{y\in Y}\{((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y)\}$ $(x \in X)$.

Let us first validate that A^* is the intuitionistic fuzzy set that lets (23) hold. Note that $(\otimes_{*1}, \to_{*1})$, $(\otimes_{*2}, \to_{*2})$ are two intuitionistic adjoint pairs. In the light of the expression of C^*, Proposition 2.18, and Proposition 2.20, it can be known that $(x \in X,\ y \in Y)$

$$\begin{aligned}&C^*(x) \le ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y),\\ &C^*(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \le B^*(y),\\ &((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \otimes_{*1} C^*(x) \le B^*(y),\\ &\alpha \otimes_{*2} (A(x) \to_{*1} B(y)) \le C^*(x) \to_{*1} B^*(y),\\ &\alpha \le (A(x) \to_{*1} B(y)) \to_{*2} (C^*(x) \to_{*1} B^*(y)).\end{aligned}$$

Furthermore, we validate that C^* is the maximum of all α-IFESI solutions for IFMT.

Suppose that C is any α-IFESI solution for IFMT. Notice that $(\otimes_{*1}, \to_{*1})$, $(\otimes_{*2}, \to_{*2})$ are two intuitionistic adjoint pairs. Then from Proposition 2.18 and Proposition 2.20, one has

$$\begin{aligned}&\alpha \le (A(x) \to_{*1} B(y)) \to_{*2} (C(x) \to_{*1} B^*(y)),\\ &\alpha \otimes_{*2} (A(x) \to_{*1} B(y)) \le C(x) \to_{*1} B^*(y),\\ &(A(x) \to_{*1} B(y)) \otimes_{*2} \alpha \le C(x) \to_{*1} B^*(y),\\ &((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \otimes_{*1} C(x) \le B^*(y),\\ &C(x) \otimes_{*1} ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \le B^*(y),\\ &C(x) \le ((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y).\end{aligned}$$

We can find that $C(x)$ is a lower bound of $\{((A(x) \to_{*1} B(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y) \mid y \in Y\}$, $x \in X$.

Then $C(x) \le C^*(x)$ $(x \in X)$ and C^* is the maximum of $\mathbb{A}_\alpha$.

Since $A^*(x) \le C^*(x)$ $(x \in X)$, then $A^* \in \mathbb{A}_\alpha$.

Finally, we further prove that A^* is the solution with maximum entropy in $\mathbb{A}_\alpha$. Suppose that A_k is any intuitionistic fuzzy set in $\mathbb{A}_\alpha$. Obviously $A_k(x) \leq C^*(x)$ $(x \in X)$.

It can be split into the following three scenarios $(x \in X)$.

(i) $(0.5, 0.5) \leq A_k(x) \leq C^*(x)$. Note that $A^*(x) = C^*(x) \wedge (0.5, 0.5)$. Here we have $A^*(x) = (0.5, 0.5)$. Then we can view it as $\pi_{A^*} \geq \tau_{A^*}$. Hence we have $\pi_{A_k} \geq \pi_{A^*},\ \tau_{A^*} \geq \tau_{A_k},\ for\ \pi_{A^*} \geq \tau_{A^*}$.

(ii) $A_k(x) \leq (0.5, 0.5) \leq C^*(x)$. Here we also have $A^*(x) = (0.5, 0.5)$. Then we can view it as $\pi_{A^*} \leq \tau_{A^*}$. So we get $\pi_{A_k} \leq \pi_{A^*},\ \tau_{A^*} \leq \tau_{A_k},\ for\ \pi_{A^*} \leq \tau_{A^*}$.

(iii) $A_k(x) \leq C^*(x) \leq (0.5, 0.5)$. Then one has $A^* = C^*$, so we can get $\pi_{A_k} \leq \pi_{A^*} \leq 0.5,\ \tau_{A_k} \geq \tau_{A^*} \geq 0.5$, then $\pi_{A^*} \leq \tau_{A^*}$. So we have $\pi_{A_k} \leq \pi_{A^*},\ \tau_{A^*} \leq \tau_{A_k},\ for\ \pi_{A^*} \leq \tau_{A^*}$.

For these three scenarios, it follows from Definition 2.12 that one has $E(A_k) \leq E(A^*)$.

In brief, A^* is the solution with maximum entropy in $\mathbb{A}_\alpha$, i.e., the formal α-IFESI solution for IFMT. □

Similar to Proposition 6.4, we get the following Proposition 7.4 from Theorem 7.3.

Proposition 7.4. *Assuming that $\rightarrow_{*1}, \rightarrow_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, and that $\otimes_{*1}$ corresponds to $\otimes_1$, and that $\rightarrow_1$, $\oplus_1$, $\ominus_1$ are associative operators of $\otimes_1$, then the formal α-IFESI solution $A^*(x) = (\pi_{A^*}(x), \tau_{A^*}(x))$ for IFMT can be expressed as $(x \in X)$:*

$$\begin{aligned}\pi_{A^*}(x) = \wedge_{y\in Y}\{[(((\pi_A(x) \rightarrow_1 \pi_B(y)) \wedge (1 - (\tau_B(y) \ominus_1 \tau_A(x)))) \otimes_{*2} \alpha_1) \rightarrow_1 \pi_{B^*}(y)] \\ \wedge [1 - (\tau_{B^*}(y) \ominus_1 ((\tau_B(y) \ominus_1 \tau_A(x)) \oplus_2 \alpha_2))]\} \wedge 0.5, \\ \tau_{A^*}(x) = \vee_{y\in Y}\{\tau_{B^*}(y) \ominus_1 ((\tau_B(y) \ominus_1 \tau_A(x)) \oplus_2 \alpha_2)\} \vee 0.5.\end{aligned} \tag{27}$$

8 Examples

Here we provide two illustrative examples to demonstrate the process of the proposed algorithm.

When we face up with n rules, (1) and (2) become:

$$\text{FMP: from n rules } A_i \rightarrow_* B_i \text{ and } A^*, \text{ compute } B^*, \tag{28}$$

$$\text{FMT: from n rules } A_i \rightarrow_* B_i \text{ and } B^*, \text{ compute } A^*, \tag{29}$$

We utilize the strategy of FITA (First-Inference-Then-Aggregation). That is, we use each $A_i \rightarrow_* B_i$ and A^* to get B_i^*; then we employ these B_i^* to get the final IFMP solution B^*. For IFMT, we employ each $A_i \rightarrow_* B_i$ and B^* to get A_i^*; then we employ these A_i^* to get the final IFMT solution A^*.

From the idea of the α-IFESI algorithm for IFMP and IFMT with multiple rules, we establish the following principles:

α-IFESI principle for IFMP with multiple rules: The result B^* of the IFMP problem is the intuitionistic fuzzy set with maximum entropy such that

$$(A_i(x) \rightarrow_{*1} B_i(y)) \rightarrow_{*2} (A^*(x) \rightarrow_{*1} B^*(y)) \geq \alpha, \tag{30}$$

holds for any $x \in X,\ y \in Y,\ i = 1, \cdots, n$, in which $\rightarrow_{*1}, \rightarrow_{*2}$ are two residual intuitionistic fuzzy implications.

α-IFESI principle for IFMT with multiple rules: The result A^* of the IFMT problem is the intuitionistic fuzzy set with maximum entropy such that

$$(A_i(x) \rightarrow_{*1} B_i(y)) \rightarrow_{*2} (A^*(x) \rightarrow_{*1} B^*(y)) \geq \alpha, \tag{31}$$

holds for any $x \in X,\ y \in Y,\ i = 1, \cdots, n$, in which $\rightarrow_{*1}, \rightarrow_{*2}$ are two residual intuitionistic fuzzy implications.

Theorem 8.1. *Assuming that $\rightarrow_{*1}, \rightarrow_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, then the formal α-IFESI solution for IFMP with FITA is as follows $(y \in Y)$:*

$$B^*(y) = \mathop{\vee}_{i=1,\cdots,n} \mathop{\vee}_{x\in X}\{A^*(x) \otimes_{*1} ((A_i(x) \rightarrow_{*1} B_i(y)) \otimes_{*2} \alpha)\} \vee (0.5, 0.5). \tag{32}$$

Proof. For each $A_i \rightarrow_* B_i$ and A^*, we get the solution $B_i^*(y) = \mathop{\vee}_{x\in X}\{A^*(x) \otimes_{*1} ((A_i(x) \rightarrow_{*1} B_i(y)) \otimes_{*2} \alpha)\} \vee (0.5, 0.5)$. Then we need that $(A_i(x) \rightarrow_{*1} B_i(y)) \rightarrow_{*2} (A^*(x) \rightarrow_{*1} B^*(y)) \geq \alpha$ holds for any $x \in X,\ y \in Y,\ i = 1, \cdots, n$. From Proposition 2.20, we know that $\rightarrow_{*1}$ is increasing w.r.t. the second variable. So we should take $B^* = \vee_{i=1}^n B_i^*$. As a result, the formal α-IFESI solution for IFMP with FITA is expressed as (32). □

Theorem 8.2. *Assuming that $\to_{*1}, \to_{*2}$ are residual intuitionistic implications respectively generated by left-continuous intuitionistic t-norms $\otimes_{*1}, \otimes_{*2}$, then the formal α-IFESI solution for IFMT with FITA is as follows ($x \in X$):*

$$A^*(x) = \bigwedge_{i=1,\cdots,n} \bigwedge_{y\in Y}\{((A_i(x) \to_{*1} B_i(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y)\} \wedge (0.5, 0.5). \tag{33}$$

Proof. For each $A_i \to_* B_i$ and B^*, we get the solution $A_i^*(x) = \bigwedge_{y\in Y}\{((A_i(x) \to_{*1} B_i(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y)\} \wedge (0.5, 0.5)$. Then we need that $(A_i(x) \to_{*1} B_i(y)) \to_{*2} (A^*(x) \to_{*1} B^*(y)) \geq \alpha$ holds for any $x \in X,\ y \in Y,\ i = 1, \cdots, n$. From Proposition 2.20, we know that $\to_{*1}$ is decreasing w.r.t. the first variable. So we should take $A^* = \wedge_{i=1}^{n} A_i^*$. As a result, the formal α-IFESI solution for IFMT with FITA is expressed as (33). □

Example 8.3. *Let $X = \{x_1, x_2, x_3, x_4, x_5\}$,$Y = \{y_1\}$, $\alpha = (0.6, 0.2)$. Suppose that $\to_{*1}$ corresponds to $\to_0$ and $\to_{*2}$ corresponds to $\to_{Lu}$ in the α-IFESI algorithm for IFMP. The rules and input are as follows:*

$A_1 = \{(0.8, 0.0), (0.1, 0.7), (0.2, 0.8), (0.2, 0.7), (0.4, 0.4)\},\ B_1 = \{(0.1, 0.7)\}$,
$A_2 = \{(0.6, 0.2), (0.4, 0.4), (0.2, 0.7), (0.3, 0.5), (0.9, 0.0)\},\ B_2 = \{(0.1, 0.7)\}$,
$A_3 = \{(0.5, 0.3), (0.6, 0.2), (0.8, 0.0), (0.7, 0.1), (0.2, 0.6)\},\ B_3 = \{(0.4, 0.4)\}$,
$A_4 = \{(0.4, 0.4), (0.5, 0.3), (0.5, 0.5), (0.5, 0.3), (0.4, 0.4)\},\ B_4 = \{(0.4, 0.4)\}$,
$A_5 = \{(0.0, 0.8), (0.7.0.1), (0.9, 0.1), (0.9, 0.0), (0.5, 0.3)\},\ B_5 = \{(0.7, 0.1)\}$,
$A_6 = \{(0.2, 0.6), (0.9, 0.0), (1.0, 0.0), (0.8, 0.2), (0.7, 0.1)\},\ B_6 = \{(0.7, 0.1)\}$,
$A^* = \{(0.6, 0.3), (0.4, 0.4), (0.6, 0.2), (0.7, 0.1), (0.3, 0.5)\}$.

This exhibits an example for fuzzy classification based upon fuzzy expert system, where three classes respectively correspond to $(0.1, 0.7), (0.4, 0.4), (0.7, 0.1)$.

From Theorem 8.1, we can get $B^*(y) = \bigvee_{i=1,\cdots,n} \bigvee_{x\in X}\{A^*(x) \otimes_{*1} ((A_i(x) \to_{*1} B_i(y)) \otimes_{*2} \alpha)\} \vee (0.5, 0.5)$ ($y \in Y$). We denote $\zeta_i(x_j, y_k) = A^*(x_j) \otimes_{*1} ((A_i(x_j) \to_{*1} B_i(y_k)) \otimes_{*2} \alpha)$.

So we can calculate that $\zeta_1(x_1, y_1) = (0.6, 0.3) \otimes_{*1} (((0.8, 0.0) \to_{*1} (0.1, 0.7)) \otimes_{*2} (0.6, 0.2)) = (0.5, 0.3) \otimes_{*1} ((0.2, 0.7) \otimes_{*2} (0.6, 0.2)) = (0.6, 0.3) \otimes_{*1} (0.0, 0.9) = (0.0, 1.0)$.

Similarly we get $\zeta_2(x_1, y_1) = (0.0, 1.0)$, $\zeta_3(x_1, y_1) = (0.0, 0.6)$, $\zeta_4(x_1, y_1) = (0.6, 0.3)$, $\zeta_5(x_1, y_1) = (0.6, 0.3)$, $\zeta_6(x_1, y_1) = (0.6, 0.3)$.

Then we have $\bigvee_{i=1,\cdots,6} \zeta_i(x_1, y_1) = (0.0, 1.0) \vee (0.0, 1.0) \vee (0.0, 0.6) \vee (0.6, 0.3) \vee (0.6, 0.3) \vee (0.6, 0.3) = (0.6, 0.3)$.

In a similar way, we obtain $\bigvee_{i=1,\cdots,6} \zeta_i(x_2, y_1) = (0.0, 0.4)$, $\bigvee_{i=1,\cdots,6} \zeta_i(x_3, y_1) = (0.0, 0.2)$, $\bigvee_{i=1,\cdots,6} \zeta_i(x_4, y_1) = (0.4, 0.2)$, $\bigvee_{i=1,\cdots,6} \zeta_i(x_5, y_1) = (0.0, 0.5)$.

Finally we achieve the formal α-IFESI solution for IFMP with FITA is $B^*(y_1) = (0.6, 0.3) \vee (0.0, 0.4) \vee (0.0, 0.2) \vee (0.4, 0.2) \vee (0.0, 0.5) \vee (0.5, 0.5) = (0.6, 0.2)$.

Because (0.6,0.2) is nearest to (0.7,0.1), the third class is what is required. □

Example 8.4. *Let $X = \{x_1\}$, $Y = \{y_1, y_2, y_3, y_4, y_5\}$, $\alpha = (0.6, 0.2)$. Suppose that $\to_{*1}$ corresponds to $\to_0$ and $\to_{*2}$ corresponds to $\to_{Lu}$ in the α-IFESI algorithm for IFMT. The rules and input are as follows:*

$A_1 = \{(0.1, 0.7)\},\ B_1 = \{(0.8, 0.0), (0.1, 0.7), (0.2, 0.8), (0.2, 0.7), (0.4, 0.4)\}$,
$A_2 = \{(0.1, 0.7)\},\ B_2 = \{(0.6, 0.2), (0.4, 0.4), (0.2, 0.7), (0.3, 0.5), (0.9, 0.0)\}$,
$A_3 = \{(0.4, 0.4)\},\ B_3 = \{(0.5, 0.3), (0.6, 0.2), (0.8, 0.0), (0.7, 0.1), (0.2, 0.6)\}$,
$A_4 = \{(0.4, 0.4)\},\ B_4 = \{(0.4, 0.4), (0.5, 0.3), (0.5, 0.5), (0.5, 0.3), (0.4, 0.4)\}$,
$A_5 = \{(0.7, 0.1)\},\ B_5 = \{(0.0, 0.8), (0.7.0.1), (0.9, 0.1), (0.9, 0.0), (0.5, 0.3)\}$,
$A_6 = \{(0.7, 0.1)\},\ B_6 = \{(0.2, 0.6), (0.9, 0.0), (1.0, 0.0), (0.8, 0.2), (0.7, 0.1)\}$,
$B^* = \{(0.3, 0.5), (0.4, 0.4), (0.2, 0.6), (0.8, 0.0), (0.5, 0.3)\}$.

This is also an example for fuzzy classification, in which three classes respectively correspond to $(0.1, 0.7)$, $(0.4, 0.4)$, $(0.7, 0.1)$.

From Theorem 8.2, we can get $A^*(x) = \bigwedge_{i=1,\cdots,n} \bigwedge_{y\in Y}\{((A_i(x) \to_{*1} B_i(y)) \otimes_{*2} \alpha) \to_{*1} B^*(y)\} \wedge (0.5, 0.5)$ ($x \in X$).

We denote $\varsigma_i(x_j, y_k) = ((A_i(x_j) \to_{*1} B_i(y_k)) \otimes_{*2} \alpha) \to_{*1} B^*(y_k)$.

So we can calculate that $\varsigma_1(x_1, y_1) = ((((0.1, 0.7) \to_{*1} (0.8, 0.0)) \otimes_{*2} (0.6, 0.2)) \to_{*1} (0.3, 0.5)) = (((1.0, 0.0) \otimes_{*2} (0.6, 0.2)) \to_{*1} (0.3, 0.5)) = ((0.6, 0.2)) \to_{*1} (0.3, 0.5)) = (0.4, 0.5)$, *Similarly we get* $\varsigma_2(x_1, y_1) = (0.4, 0.5)$, $\varsigma_3(x_1, y_1) = (0.4, 0.5)$, $\varsigma_4(x_1, y_1) = (0.4, 0.5)$, $\varsigma_5(x_1, y_1) = (1.0, 0.0)$, $\varsigma_6(x_1, y_1) = (1.0, 0.0)$.

Then we get $\bigvee_{i=1,\cdots,6} \varsigma_i(x_1, y_1) = (0.4, 0.5) \wedge (0.4, 0.5) \wedge (0.4, 0.5) \wedge (0.4, 0.5) \wedge (1.0, 0.0) \wedge (1.0, 0.0) = (0.4, 0.5)$.

In a similar mode, we obtain $\bigvee_{i=1,\cdots,6} \varsigma_i(x_1, y_2) = (0.4, 0.4)$, $\bigvee_{i=1,\cdots,6} \varsigma_i(x_1, y_3) = (0.4, 0.6)$, $\bigvee_{i=1,\cdots,6} \varsigma_i(x_1, y_4) = (1.0, 0.0)$, $\bigvee_{i=1,\cdots,6} \varsigma_i(x_1, y_5) = (0.4, 0.5)$.

Together we achieve the formal α-IFESI solution for IFMT with FITA is as follows: $A^*(x_1) = (0.4, 0.5) \wedge (0.4, 0.4) \wedge (0.4, 0.6) \wedge (1.0, 0.0) \wedge (0.4, 0.5) \wedge (0.5, 0.5) = (0.4, 0.6)$. *Because (0.4,0.6) is nearest to (0.4,0.4), the second class is what is required.* □

9 Discussions

The special properties of the proposed IFESI algorithms are mainly reflected in three ways.

First of all, the idea of the proposed IFESI algorithms is special. Fuzzy reasoning should consider the factors of both the logic system and the inference model. In (16), $\rightarrow_{*1}$ reflects the implication connective in a logic system, while $\rightarrow_{*2}$ characterizes the "if-then" relation of fuzzy inference model "if A implies B, then A^* implies B^*". Moreover, aiming at the environment of intuitionistic fuzzy sets, we synthetically take the maximum fuzzy entropy, the logic system and the reasoning model into consideration, and then the IFESI algorithms are derived.

Moreover, the IFESI algorithm is C-reductive for IFMP, in which

$$C\ means\ (P25) + (P26).$$

Meanwhile, the IFESI algorithm is C-reductive for IFMT, where

$$C\ means\ (P27) + (P28) + (P29).$$

Lastly, if the t-norm $\otimes_1$ is continuous, then the IFESI algorithm of IFMP is continuous and uniformly continuous in NHD. In the meantime, if $\rightarrow_1$ is continuous w.r.t. the second variable, then the IFESI algorithm of IFMT is continuous and uniformly continuous in NHD.

Here we show some practical applications of the proposed IFESI algorithms.

To begin with, the proposed IFESI algorithms can be applied to fuzzy classification based on fuzzy expert systems. Actually, Examples 8.3 and 8.4 are the examples of fuzzy classification. Many classification problems in reality can be solved in this way.

Furthermore, as an important branch of affective computing, emotion deduction (which discovers how to generate proper values of other emotions from some basic emotions) is also an important application point for the IFESI algorithms. The values of basic emotions are regarded as A_i and the values of other emotions are modeled by B_i in (28) under the environment of intuitionistic fuzzy sets. Then we can realize the emotion deduction process via the IFESI algorithms.

Finally, the IFESI algorithms can also be used to build fuzzy controllers, which incorporate fuzzifier, fuzzy reasoning method, and defuzzifier. In detail, we employ the IFESI algorithms to carry on fuzzy reasoning, and select appropriate fuzzifier and defuzzifier. Then we can establish corresponding fuzzy controllers. Through these fuzzy controllers, there can be a wider range of practical applications.

In fact, there are many other practical applications however they are not listed here.

10 Conclusions and Prospect

In this study, we come up with the IFESI algorithm, and then extend it to the α-version. The main contributions and conclusions are outlined as follows.

(i) New symmetric implicational principles are presented. Then the unified solutions of the IFESI algorithm are obtained for IFMP and IFMT, which are based on residual intuitionistic implications.

(ii) The reductive properties of the IFESI algorithm are validated for IFMP and IFMT.

(iii) The continuity of the IFESI algorithm is analyzed for IFMP and IFMT.

(iv) The IFESI algorithm is extended into the α-IFESI algorithm, and the unified solutions of the α-IFESI algorithm are obtained for IFMP and IFMT.

(v) Two examples of fuzzy classification for the α-IFESI algorithm are provided to reveal the detailed computing process of the IFESI algorithm.

In the future, it would be of interest to investigate the fuzzy controllers based on the proposed fuzzy reasoning algorithm, and analyze corresponding performance.

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