Oscillation-Bound Estimation of Perturbations Under Bandler–Kohout Subproduct

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***Abstract*—The Bandler–Kohout subproduct (BKS) method is one of the two widely acknowledged fuzzy relational inference (FRI) schemes. The previous works related to its stability and robustness mainly concentrated on how the output values were changed with perturbation parameters of input values. However, the works on estimating oscillation bounds of output values with regard to varying limits of input, are lacking. In this study, we investigate the oscillation-bound estimation of perturbations for BKS. First, the BKS output variation scopes are acquired for interval perturbation, where the *R*-implication, (*S*, *N*)-implication, QL-implication, and *t*-norm implication are adopted. Second, in allusion to the more sophisticated problem of the fuzzy reasoning chain with BKS, the oscillation bounds of BKS output resulting from input interval perturbation are offered. Third, we construct the upper and lower bounds of BKS output deviation originated in the simple perturbation of the input fuzzy set, in which the situations of one rule and multiple rules are both dissected. Finally, the stable properties of all these BKS strategies are confirmed. It is emphasized that interval perturbation and simple perturbation are more general ways to give expression describing the robustness issue, and the obtained oscillation bounds also deliver more detailed characterization of the output deviation along with the input perturbation. This study further validates the sound properties of the BKS method.**

***Index Terms*—Bandler–Kohout subproduct (BKS), compositional rule of inference (CRI), fuzzy reasoning, perturbation, robustness, stability.**

I. Introduction

FUZZY reasoning is embodied as an advanced computing framework on the strength of the concepts of the fuzzy set, fuzzy if-then rule, and approximate reasoning. It has been thoroughly explored and applied to the fields of fuzzy control, pattern recognition, decision making, time-series analysis and others. The general form of fuzzy reasoning comes as the following fuzzy modus ponens (FMP):

$$\begin{array}{ll} \text{Input:} & u \text{ is } C \\ \underline{\text{Rule:}} & \underline{\text{If } u \text{ is } A, \text{ then } v \text{ is } B} \\ \text{Output:} & D \text{ is } f(A, B, C) = ? \end{array} \tag{1}$$

in which $A, C \in F(U)$, $B, D \in F(V)$, and $F(U)$, $F(V)$, respectively, represent the set of all fuzzy subsets of universes U and V. f signifies a fuzzy reasoning mechanism. The output $D = f(A, B, C)$ is obtained from the rule and the input with the fuzzy reasoning mechanism f.

Directing at such a fuzzy reasoning problem, there are a number of reasoning mechanisms in fuzzy rule-based systems, including fuzzy relational inference (FRI) [1], similarity-based reasoning (SBR) [2], and inverse truth functional modification [3], fuzzy interpolative reasoning, and differently implicational reasoning [5], as a few representatives. Among them, FRI has gained visible attention from both theoretical and practical points of view. Among the FRI methods, the two most frequently considered strategies are the compositional rule of inference (CRI) [6] established by Zadeh, and the Bandler–Kohout subproduct (BKS) proposed by our research team [7] (realized on the basis of the previous results of Bandler and Kohout [8]). In this work, we concentrate on the FRI mechanism, especially the BKS method.

A. BKS and CRI

Here, we briefly review the BKS method. Our research team [7] came up with the BKS method of fuzzy reasoning. In allusion to a given SISO rule in (1), the BKS reasoning mode is shown as follows:

$$D = C \lhd R. \tag{2}$$

Thereinto, $C \in F(U)$ is the input. $R : U \times V \to [0, 1]$ means the fuzzy relation, viz., $R \in F(U \times V)$ denotes the rule base. $D \in F(V)$ represents the constructed output. In the end, the operation $\lhd$ is the mapping $\lhd : F(U) \times F(U \times V) \to F(V)$ conveyed by

$$D(v) = \inf_{u \in U} \{C(u) \to R(u, v)\}, \quad v \in V. \tag{3}$$

Here, $\to$ represents a fuzzy implication. The mapping $\lhd$ is also known as the $\inf -I$ composition, where I indicates a

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fuzzy implication. For the rule base R, two alternatives are sought

$$\check{R}(u, v) = \vee_{i=1}^{n}\{A_i(u) \otimes B_i(v)\}, \quad u \in U, \quad v \in V \tag{4}$$

$$\hat{R}(u, v) = \wedge_{i=1}^{n}\{A_i(u) \to B_i(v)\}, \quad u \in U, \quad v \in V. \tag{5}$$

Here, $\otimes$ denotes a t-norm, and $\to$ represents a fuzzy implication. Equation (4) corresponds to a conjunctive mode for the rule base, while (5) denotes an implicative mode. As for the difference in semantics expressed by $\check{R}$ and $\hat{R}$, refer to [9]. By making use of $\check{R}$ or $\hat{R}$, the corresponding BKS algorithm becomes

$$D(v) = \inf_{u \in U}\left\{C(u) \to \vee_{i=1}^{n}(A_i(u) \otimes B_i(v))\right\}, \quad v \in V \tag{6}$$

$$D(v) = \inf_{u \in U}\left\{C(u) \to \wedge_{i=1}^{n}(A_i(u) \to B_i(v))\right\}, \quad v \in V. \tag{7}$$

For the case of a single rule, we can acquire two types of the BKS methods

$$D(v) = \inf_{u \in U}\{C(u) \to (A(u) \otimes B(v))\}, \quad v \in V \tag{8}$$

$$D(v) = \inf_{u \in U}\{C(u) \to (A(u) \to B(v))\}, \quad v \in V. \tag{9}$$

The BKS methods indicated by (6) or (8) are referred to as the BKS-T methods, meanwhile, the BKS methods completed with the aid of (7) or (9) are said to be the BKS-I methods.

Another classic fuzzy reasoning method is the CRI method proposed by Zadeh [6]. The CRI method can be shown as follows:

$$D(v) = \sup_{u \in U}\{C(u) \otimes R(u, v)\}, \quad v \in V \tag{10}$$

where $\otimes$ is a fuzzy conjunction, which is usually treated as a t-norm (see [10]).

Here, we compare the CRI method with the BKS method. Notice that $\hat{R}$ and $\check{R}$ are typical and recognized fuzzy relations used to model the rule base. On the one hand, any fuzzy rule in $\hat{R}$ is considered as a constraint, which results in a conjunctive way of integrating the individual constraints, because of more constraints, the fewer possible values satisfying these constraints [9], [11]. The employed fuzzy implication obviously expresses the logical operation IF-THEN while the minimum evidently reflects the logical conjunction, and hence $\hat{R}$ is an appropriate structure of conjunctively merged IF-THEN fuzzy rules. On the other hand, $\check{R}$ is not mainly constructed as a mathematical structure of conditional IF-THEN sentences but it adopts the Cartesian product of antecedent and consequent fuzzy sets. Fuzzy rules characterized by $\check{R}$ are regarded as pieces of data that are evidently accumulated by the maximum in (4) for the merging of fuzzy rules [9], [11]. It is emphasized that some of the superiorities of the CRI method depend largely on the fuzzy rules modeled by $\check{R}$. However, aiming at $\hat{R}$, many of the advantages of CRI have been lost, which incorporate the robustness of CRI in regard to the indistinguishability of input fuzzy sets in the situation of multiple rules, the equivalence of first-infer-then-aggregate (FITA), and first-aggregate-then-infer (FATI), and many strategies to improve the efficiency of the reasoning process [12]. But the BKS method has ideal performance for the fuzzy rules modeled by $\hat{R}$ [12]. Generally speaking, in view of previous research (e.g., [12]–[14]), when a given fuzzy rule base is modeled by the fuzzy relation $\check{R}$, the CRI method is preferable; however, when it is modeled by the fuzzy relation $\hat{R}$, the BKS method is better. In addition, the fuzzy relation $\hat{R}$ does better in handling different levels of specificity of knowledge than $\check{R}$, and only $\hat{R}$ has a built-in ability for detecting inconsistencies in rule bases [15]. Therefore, for complex tasks, such as in data mining, logic programming, or complex expert systems, the combination of BKS with $\hat{R}$ provides a better selection than the combination of CRI with $\check{R}$.

B. Research on Stability and Robustness for CRI and BKS

For a fuzzy reasoning mechanism, its stability and robustness are deemed as the key research topics. At present, fuzzy reasoning has become a vital strategy for designing and analyzing fuzzy controllers. The fuzzy controllers perform well when using the quantitative expression to characterize granular domain knowledge, thus the divergence of the latter from related quantitative representations enhances the importance of the stability issue for fuzzy controllers [16]. The stability of fuzzy reasoning concerns the deviation of the output implied by perturbations of the input. Current research chiefly focuses on how the output values of the reasoning strategy are changed in response to the perturbation parameters of input values.

In this regard, some studies were completed, especially for the CRI method. Pappis [17] established the proximity measure originated from approximately equal fuzzy values of system variables and relations and revealed its retention abilities in the setting of the CRI method. Then, Hong and Hwang [18] extended it to the α-similarity measure. Ying [19] presented maximum and average perturbations of fuzzy sets, determined perturbation parameters for some CRI strategies. Cai [20] made a thorough inquiry into the robustness of specific fuzzy implications and reasoning rules in CRI strategies with regard to δ-equalities of fuzzy sets. Jin *et al.* [21] offered the concept of perturbation of fuzzy sets with a logic-oriented equivalence measure and obtained corresponding robustness outcomes for some fuzzy-logic connectives, fuzzy implications, reasoning rules, and fuzzy reasoning machines. Dai *et al.* [22] investigated the perturbation of CRI with respect to the perturbation of fuzzy sets derived from normalized Minkowski distances. Li *et al.* [23] adopted an idea similar to the modulus of continuity to analyze the robustness of some fuzzy connectives together with the CRI method. Li *et al.* [24] proposed a divergence measure on the strength of previously proposed dissimilarity function DF-metric [25] and then inquired into the robustness of fuzzy connectives and CRI. As for the BKS algorithm, there are also some works related to this aspect. Štěpnička and Jayaram[12] verified the effect of the BKS method with residuated implications, which included the equivalence and reasonability for the solvability and interpolativity, and analyzed its robustness (i.e., the preservation of the indistinguishability that may be inherent in the input fuzzy sets) along analogous lines of study as [26] for CRI. Finally, it was concluded that the BKS strategy is a sound alternative as the CRI method. Henceforth, Mandal and Jayaram [13] investigated the BKS method employing Yager's classes of fuzzy implications and found that the corresponding BKS method also has useful properties, viz., interpolativity, continuity, and robustness (whose definitions are analogous to those present in [12]). Štěpnička *et al.* [27] verified the robustness with regard to the combination of BKS and implicative model from another viewpoint, and briefly analyzed the preservation of desirable properties for FRI methods. Focusing on the satisfiability problem for the three "interpolation" axioms introduced by Moser and Navara in [28], Štěpnička and Mandal [14] further developed these axioms and discovered the corresponding conditionally firing rules as well as the related properties for the CRI and BKS algorithms with Mamdani–Assilian and implicative models.

As a matter of fact, there is another aspect of the stability of fuzzy reasoning, that is, to research how to estimate oscillation bounds of output values when variation limits of input values are provided. In [29], in allusion to the CRI method with a single rule, Cheng and Fu analyzed the upper and lower bounds of the output error induced by the simple perturbation of the input fuzzy set and determined the oscillation scope of the output result for the case of which the input ranges over an interval.

C. Motivation

It should be emphasized that the oscillation-bound estimation for BKS demands to be explored.

The previous investigation on error estimation for CRI can also be done for other FRI mechanisms. In this article, we focus on the oscillation-bound estimation for the BKS algorithm.

Some properties of the CRI mechanism rely, to a great extent, on the modeling strategy used for fuzzy rules via the fuzzy relation $\check{R}$, which is proper in the context in which the fuzzy rules are regarded as positive pieces of information [9], [11]. For example, the work [29] concerns the case of a single rule with $\check{R}$ formed in CRI [viz., $R(u, v) = A(u) \to B(v)$]. However, there are also situations in which the conditional aspect of fuzzy rules is considered, and the fuzzy relation $\hat{R}$ has to be employed to characterize them [15]. Hence, some properties of CRI cannot hold, including the oscillation-bound estimation. As a consequence, the oscillation-bound estimation of BKS has to be formed for both $\check{R}$ and $\hat{R}$.

D. Outline

Section II covers some preliminaries. Section III discusses the estimation of the limits of the output with regard to the interval perturbation of the input. Section IV analyzes the estimates of the output scope related to the interval perturbation of the input in the problem of a chain of fuzzy reasoning via BKS. Section V investigates the error estimation of perturbations for BKS with a single rule and with multiple rules. The stability of the BKS fuzzy reasoning method is validated. Section VI offers some discussion. Section VII summarizes this article.

II. Preliminaries

A. Some Notions

There are several definitions of fuzzy implications. The fundamental one is as follows.

Definition 1 [8], [30]: A fuzzy implication on $[0, 1]$ is a function $I : [0, 1]^2 \to [0, 1]$ satisfying the condition

$$\text{(P1)}\quad I(0, 0) = I(0, 1) = I(1, 1) = 1,\ I(1, 0) = 0.$$

$I(a, b)$ can also be be represented $a \to b$ $(a, b \in [0, 1])$.

Definition 2 [10]: A function $\otimes : [0, 1]^2 \longrightarrow [0, 1]$ is referred to as a t-norm if $\otimes$ is associative, commutative, increasing, and lets $1 \otimes p = p$ be valid $(p \in [0, 1])$.

Here, three commonly used t-norms are adopted, which incorporate min $\otimes_M$, product $\otimes_P$, and the Lukasiewicz conjunction $\otimes_L$. That is, $\otimes_M(p, q) = p \wedge q$, $\otimes_P(p, q) = p \times q$, and $\otimes_L(p, q) = 0 \vee (p + q - 1)$ where $p, q \in [0, 1]$ and $\vee$ and $\wedge$ signify the supremum and infimum in turn.

Definition 3 [10]: A function $\oplus : [0, 1]^2 \longrightarrow [0, 1]$ goes by the name of a t-conorm if $\oplus$ is associative, commutative, increasing, and makes $0 \oplus p = p$ be effective $(p \in [0, 1])$.

Definition 4 [10]: A fuzzy negation is a decreasing function $N : [0, 1] \longrightarrow [0, 1]$ which meets $N(0) = 1$ and $N(1) = 0$.

Definition 5 [30]: A fuzzy implication $\to$ is referred to as an R-implication whenever a left-continuous t-norm $\otimes$ exists and lets $p \to q = \vee\{x \in [0, 1] | p \otimes x \le q\}$ be right $(p, q \in [0, 1])$.

Definition 6 [31]: A function $I : [0, 1]^2 \to [0, 1]$ is known as an (S, N)-implication whenever there exist a t-conorm $\oplus$ and a fuzzy negation N making $I(p, q) = N(p) \oplus q$ hold $(p, q \in [0, 1])$. Such (S, N)-implication is recorded as $I_{\oplus,N}$.

Definition 7 [30], [32]: A function $I : [0, 1]^2 \to [0, 1]$ is referred to as a QL-implication whenever there are a t-norm $\otimes$, a t-conorm $\oplus$, and a fuzzy negation N making $I(p, q) = N(p) \oplus (p \otimes q)$ $(p, q \in [0, 1])$ be effective. Such QL-implication is represented by $I_{\otimes,\oplus,N}$.

Definition 8 [29], [33]: A function $I : [0, 1]^2 \to [0, 1]$ is referred to as a t-norm implication whenever there exists a t-norm $\otimes$ making $I(p, q) = p \otimes q$ be right $(p, q \in [0, 1])$.

Although the t-norm implications do not meet (P1), they are adopted as a model of fuzzy implication in many applications of fuzzy logic [33].

Definition 9 [17]: Suppose that $C, D \in F(Z)$ and that $\lambda \in [0, 1]$. If $\sup_{z \in Z} |C(z) - D(z)| \le \lambda$ holds, then C and D are said to be approximately equal. λ is known as a proximity measure of C and D.

Definition 10 [18]: Suppose that $C, D \in F(Z)$ and that $\lambda \in [0, 1]$. If $1 - \sup_{z \in Z} |C(z) - D(z)| \ge \lambda$ holds, then C and D are said to be λ-similar.

Definition 11 [19]: Suppose that $C, D \in F(Z)$ and that $\lambda \in [0, 1]$. If one has $|C(z) - D(z)| \le \lambda$ for any $z \in Z$, viz., $D(z) = C(z) + \lambda\psi(z)$ for some function $\psi : Z \to [-1, 1]$ (where $0 \le C(z) + \lambda\psi(z) \le 1$ for any $z \in Z$), then D is referred to as a maximum λ perturbation of C.

Definition 12 [34]: Suppose that $C, D \in F(Z)$ and that $\lambda \in [0, 1]$. If $\sup_{z \in Z} |C(z) - D(z)| \le 1 - \lambda$ holds, then C and D are called λ-equal.

Definition 13 [29]:

1) Suppose that $\beta^-, \beta^+ \in F(Z)$ and that $\beta^-(z) \le \beta^+(z)$ $(z \in Z)$, then $[\beta^-, \beta^+]$ is known as a fuzzy interval on Z.
2) Suppose that $C \in F(Z)$ and that $[\beta^-, \beta^+]$ be a fuzzy interval on Z. If $\beta^-(z) \le C(z) \le \beta^+(z)$ $(z \in Z)$, then

$[\beta^-, \beta^+]$ is referred to as an interval perturbation of C, which is signified by $C \in [\beta^-, \beta^+]$.

Definition 14 [29]: Suppose that $A, B \in F(Z)$. If there is a function $\beta : Z \to [-1, 1]$ making $B(z) = A(z) + \beta(z)$ $(z \in Z)$, then B is known as a simple perturbation of A, and β is called a perturbation factor of A.

Definition 15 [29]: Assume that $C, E \in F(Z)$, $D \in F(W)$, and that $C \in [\alpha^-, \alpha^+]$, $D \in [\eta^-, \eta^+]$, $E \in [\gamma^-, \gamma^+]$. A fuzzy reasoning strategy f is referred to as a stable function if, for any $\varepsilon > 0$, there is a fuzzy interval $[\lambda^-, \lambda^+]$ on W and $\delta > 0$ making $\lambda^+(v) - \lambda^-(v) < \varepsilon$ be effective for any $w \in W$ and $f(C, D, E) \in [\lambda^-, \lambda^+]$ if $\alpha^+(z) - \alpha^-(z) < \delta$, $\eta^+(w) - \eta^-(w) < \delta$, $\gamma^+(z) - \gamma^-(z) < \delta$ $(z \in Z,\ w \in W)$.

Definition 16 [29]:

1) Assume that $C, E \in F(Z)$, $D \in F(W)$, and that C^*, D^*, E^* are perturbations of C, D, E with factors $\beta_1, \beta_2, \beta_3$ in turn. A fuzzy reasoning strategy f is known as a stable function if, for any $\varepsilon > 0$, there is $\delta > 0$ making $|f(C^*, D^*, E^*)(w) - f(C, D, E)(w)| < \varepsilon$ hold if $|\beta_1(z)| < \delta$, $|\beta_2(w)| < \delta$, $|\beta_3(z)| < \delta$ $(z \in Z,\ w \in W)$.
2) Assume that $\{C^*_m, D^*_m, E^*_m\}$ is a perturbation sequence of (C, D, E) with regard to $\{\beta_{1m}, \beta_{2m}, \beta_{3m}\}$ of factors $(m = 1, 2, \ldots)$. Then, a fuzzy reasoning strategy f is referred to as an asymptotic stable function if, for any $\varepsilon > 0$, there are m_0 and $\delta > 0$ making $|f(C^*_m, D^*_m, E^*_m)(w) - f(C, D, E)(w)| < \varepsilon$ hold for any $m > m_0$ if $|\beta_{1m}(z)| < \delta$, $|\beta_{2m}(w)| < \delta$, and $|\beta_{3m}(z)| < \delta$ $(z \in Z,\ w \in W)$.

Lemma 1 [29]: Let h_1 and h_2 be real-valued, bounded mappings on Z, and $C, D \in F(Z)$. Thus, one has:

1) $h_1 \vee h_2 = \max\{h_1, h_2\} = (h_1 + h_2)/2 + |h_1 - h_2|/2$;
2) $h_1 \wedge h_2 = \min\{h_1, h_2\} = (h_1 + h_2)/2 - |h_1 - h_2|/2$;
3) $-|h_1 - h_2| \le |h_1| - |h_2| \le |h_1 - h_2|$;
4) $\inf_{z\in Z} C(z) + \inf_{z\in Z} D(z) \le \inf_{z\in Z}(C(z) + D(z))$;
5) $\sup_{z\in Z}(C(z) + D(z)) \le \sup_{z\in Z} C(z) + \sup_{z\in Z} D(z)$;
6) $\sup_{z\in Z} C(z) = -\inf_{z\in Z}(-C(z))$;
7) $\inf_{z\in Z}(C(z) \wedge D(z)) = \inf_{z\in Z} C(z) \wedge \inf_{z\in Z} D(z)$;
8) $\inf_{z\in Z}(C(z) \vee D(z)) \ge \inf_{z\in Z} C(z) \vee \inf_{z\in Z} D(z)$;
9) $\sup_{z\in Z}(C(z) \wedge D(z)) \le \sup_{z\in Z} C(z) \wedge \sup_{z\in Z} D(z)$;
10) $\sup_{z\in Z}(C(z) \vee D(z)) = \sup_{z\in Z} C(z) \vee \sup_{z\in Z} D(z)$.

It is effortless to verify Lemma 2.

Lemma 2: If h_1 and h_2 be real-valued, bounded mappings on Z, then:

1) $\sup_{z\in Z} h_1(z) - \sup_{z\in Z} h_2(z) \le \sup_{z\in Z}(h_1(z) - h_2(z))$;
2) $\inf_{z\in Z} h_1(z) - \inf_{z\in Z} h_2(z) \le \sup_{z\in Z}(h_1(z) - h_2(z))$;
3) $\inf_{z\in Z} h_1(z) - \inf_{z\in Z} h_2(z) \ge \inf_{z\in Z}(h_1(z) - h_2(z))$;
4) $\sup_{z\in Z} h_1(z) - \sup_{z\in Z} h_2(z) \ge \inf_{z\in Z}(h_1(z) - h_2(z))$.

III. Interval Perturbation for BKS

A. Interval Perturbation for BKS With Single Rule

Assume that $[\alpha^-, \alpha^+]$ and $[\gamma^-, \gamma^+]$ are fuzzy intervals on U, and that $[\eta^-, \eta^+]$ is a fuzzy interval on V. We denote $\tilde{i}_u(f) = \inf_{u\in U}\{f(u)\}$, $\tilde{s}_u(f) = \sup_{u\in U}\{f(u)\}$, and $BKS_1(v) = BKS(A, B, C)(v)$ $(v \in V)$.

Theorem 1: Let $A, C \in F(U)$ and $B \in F(V)$. Assume that $A \in [\alpha^-, \alpha^+]$, $C \in [\gamma^-, \gamma^+]$, and $B \in [\eta^-, \eta^+]$, and that $\otimes$ is a t-norm. The BKS-T method is adopted.

1) If $\to$ is an R-implication, then $\tilde{s}_u(\gamma^+) \lhd (\tilde{i}_u(\alpha^-) \otimes \eta^-(v)) \le BKS_1(v) \le \tilde{i}_u(\gamma^-) \lhd (\tilde{s}_u(\alpha^+) \otimes \eta^+(v))$.
2) If $\to$ is a t-norm implication, then $\tilde{i}_u(\gamma^-) \lhd (\tilde{i}_u(\alpha^-) \otimes \eta^-(v)) \le BKS_1(v) \le \tilde{s}_u(\gamma^+) \lhd (\tilde{s}_u(\alpha^+) \otimes \eta^+(v))$.
3) If $\to$ is an (S, N)-implication $I_{\oplus_1,N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 (\tilde{i}_u(\alpha^-) \otimes \eta^-(v))\} \le BKS_1(v) \le \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 (\tilde{s}_u(\alpha^+) \otimes \eta^+(v))\}$.
4) If $\to$ is an QL-implication $I_{\otimes_1,\oplus_1,N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 [\tilde{i}_u(\gamma^-) \otimes_1 (\tilde{i}_u(\alpha^-) \otimes \eta^-(v))]\} \le BKS_1(v) \le \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 [\tilde{s}_u(\gamma^+) \otimes_1 (\tilde{s}_u(\alpha^+) \otimes \eta^+(v))]\}$.

Proof: Here, we only prove 1). The other cases can be gained in an analogous manner. Note that an R-implication is decreasing in the first variable and increasing in the second variable. From the intrinsic characteristics of R-implications and t-norms, one has $\tilde{s}_u(\gamma^+) \lhd (\tilde{i}_u(\alpha^-) \otimes \eta^-(v)) = \inf_{u\in U}\{\sup_{u\in U}(\gamma^+(u)) \to (\inf_{u\in U}(\alpha^-(u)) \otimes \eta^-(v))\} \le BKS_1(v) = \inf_{u\in U}\{C(u) \to (A(u) \otimes B(v))\} \le \inf_{u\in U}\{\inf_{u\in U}(\gamma^-(u)) \to (\sup_{u\in U}(\alpha^+(u)) \otimes \eta^+(v))\} = \tilde{i}_u(\gamma^-) \lhd (\tilde{s}_u(\alpha^+) \otimes \eta^+(v))$. ■

Theorem 2 can be derived in a similar manner.

Theorem 2: Let $A, C \in F(U)$ and $B \in F(V)$. Assume that $A \in [\alpha^-, \alpha^+]$, and $C \in [\gamma^-, \gamma^+]$, $B \in [\eta^-, \eta^+]$. The BKS-I method is used.

1) If $\to$ is an R-implication, then $\tilde{s}_u(\gamma^+) \lhd (\tilde{s}_u(\alpha^+) \to \eta^-(v)) \le BKS_1(v) \le \tilde{i}_u(\gamma^-) \lhd (\tilde{i}_u(\alpha^-) \to \eta^+(v))$.
2) If $\to$ is a t-norm implication, then $\tilde{i}_u(\gamma^-) \lhd (\tilde{i}_u(\alpha^-) \to \eta^-(v)) \le BKS_1(v) \le \tilde{s}_u(\gamma^+) \lhd (\tilde{s}_u(\alpha^+) \to \eta^+(v))$.
3) If $\to$ is an (S, N)-implication $I_{\oplus_1,N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 (N(\tilde{s}_u(\alpha^+)) \oplus_1 \eta^-(v))\} \le BKS_1(v) \le \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 (N(\tilde{i}_u(\alpha^-)) \oplus_1 \eta^+(v))\}$.
4) If $\to$ is an QL-implication $I_{\otimes_1,\oplus_1,N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 [\tilde{i}_u(\gamma^-) \otimes_1 (N(\tilde{s}_u(\alpha^+)) \oplus_1 (\tilde{i}_u(\alpha^-) \otimes_1 \eta^-(v)))]\} \le BKS_1(v) \le \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 [\tilde{s}_u(\gamma^+) \otimes_1 (N(\tilde{i}_u(\alpha^-)) \oplus_1 (\tilde{s}_u(\alpha^+) \otimes_1 \eta^+(v)))]\}$.

If the generic operations (including t-norm, t-conorm, and fuzzy negation) are continuous, the BKS method with $\otimes$ (some t-norm) and an (S, N)-implication, or a QL-implication, or a t-norm implication is stable. As for the situation of R-implication, if the R-implication and t-norm are continuous, then the corresponding BKS fuzzy reasoning method is stable.

Suppose that there is a perturbation sequence $([\alpha^-_m, \alpha^+_m], [\eta^-_m, \eta^+_m], [\gamma^-_m, \gamma^+_m])$ of the input (A, B, C) for the BKS method implying that $\lim_{m\to\infty} \sup_{u\in U}(\alpha^+_m(u) - \alpha^-_m(u)) = \lim_{m\to\infty} \sup_{v\in V}(\eta^+_m(v) - \eta^-_m(v)) = \lim_{m\to\infty} \sup_{u\in U}(\gamma^+_m(u) - \gamma^-_m(u)) = 0$ hold $(m = 1, 2, \ldots)$. In Theorems 1 and 2, we adopt λ^-_m, λ^+_m to indicate the corresponding lower and upper bounds of the BKS output, viz., $BKS(A, B, C) \in [\lambda^-_m, \lambda^+_m]$ for $([\alpha^-_m, \alpha^+_m], [\eta^-_m, \eta^+_m], [\gamma^-_m, \gamma^+_m])$ $(m = 1, 2, \ldots)$. When the continuous condition are met, one has from Theorems 1 and 2 that $\lim_{m\to\infty} \sup_{v\in V}(\lambda^+_m(v) - \lambda^-_m(v)) = 0$. That is, the output of BKS algorithm converges steadily to a value when the continuous conditions mentioned above is effective. From Definition 15, the BKS algorithm for the situation of one rule is stable from the perspective of interval perturbation.

B. Interval Perturbation for BKS With Multiple Rules

Denote $BKS_n(v) = BKS(A_1, \ldots, A_n, B_1, \ldots, B_n, C)(v)$ ($v \in V$). Theorems 3 and 4 can be proved analogously.

Theorem 3: Suppose that $A_i, C \in F(U)$, $B_i \in F(V)$, and that $A_i \in [\alpha_i^-, \alpha_i^+]$, and $C \in [\gamma^-, \gamma^+]$, $B_i \in [\eta_i^-, \eta_i^+]$, and that $\otimes$ is a t-norm ($i = 1, 2, \ldots, n$). The BKS-T method is adopted.

1) If $\rightarrow$ is an R-implication, then $\tilde{s}_u(\gamma^+) \lhd [\vee_{i=1}^n (\tilde{i}_u(\alpha_i^-) \otimes \eta_i^-(v))] \leq BKS_n(v) \leq \tilde{i}_u(\gamma^-) \lhd [\vee_{i=1}^n (\tilde{s}_u(\alpha_i^+) \otimes \eta_i^+(v))]$.
2) If $\rightarrow$ is a t-norm implication, then $\tilde{i}_u(\gamma^-) \lhd [\vee_{i=1}^n (\tilde{i}_u(\alpha_i^-) \otimes \eta_i^-(v))] \leq BKS_n(v) \leq \tilde{s}_u(\gamma^+) \lhd [\vee_{i=1}^n (\tilde{s}_u(\alpha_i^+) \otimes \eta_i^+(v))]$.
3) If $\rightarrow$ is an (S, N)-implication $I_{\oplus 1, N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 \vee_{i=1}^n (\tilde{i}_u(\alpha_i^-) \otimes \eta_i^-(v))\} \leq BKS_n(v) \leq \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 \vee_{i=1}^n (\tilde{s}_u(\alpha_i^+) \otimes \eta_i^+(v))\}$.
4) If $\rightarrow$ is a QL-implication $I_{\otimes 1, \oplus 1, N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 [\tilde{i}_u(\gamma^-) \otimes_1 \vee_{i=1}^n (\tilde{i}_u(\alpha_i^-) \otimes \eta_i^-(v))]\} \leq BKS_n(v) \leq \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 [\tilde{s}_u(\gamma^+) \otimes_1 \vee_{i=1}^n (\tilde{s}_u(\alpha_i^+) \otimes \eta_i^+(v))]\}$.

Theorem 4: Suppose that $A_i, C \in F(U)$, $B_i \in F(V)$, and that $A_i \in [\alpha_i^-, \alpha_i^+]$, $C \in [\gamma^-, \gamma^+]$, and $B_i \in [\eta_i^-, \eta_i^+]$ ($i = 1, 2, \ldots, n$). The BKS-I method is utilized.

1) If $\rightarrow$ is an R-implication, then $\tilde{s}_u(\gamma^+) \lhd [\wedge_{i=1}^n (\tilde{s}_u(\alpha_i^+) \rightarrow \eta_i^-(v))] \leq BKS_n(v) \leq \tilde{i}_u(\gamma^-) \lhd [\wedge_{i=1}^n (\tilde{i}_u(\alpha_i^-) \rightarrow \eta_i^+(v))]$.
2) If $\rightarrow$ is a t-norm implication, then $\tilde{i}_u(\gamma^-) \lhd [\wedge_{i=1}^n (\tilde{i}_u(\alpha_i^-) \rightarrow \eta_i^-(v))] \leq BKS_n(v) \leq \tilde{s}_u(\gamma^+) \lhd [\wedge_{i=1}^n (\tilde{s}_u(\alpha_i^+) \rightarrow \eta_i^+(v))]$.
3) If $\rightarrow$ is an (S, N)-implication $I_{\oplus 1, N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 \wedge_{i=1}^n (N(\tilde{s}_u(\alpha_i^+)) \oplus_1 \eta_i^-(v))\} \leq BKS_n(v) \leq \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 \wedge_{i=1}^n (N(\tilde{i}_u(\alpha_i^-)) \oplus_1 \eta_i^+(v))\}$.
4) If $\rightarrow$ is an QL-implication $I_{\otimes 1, \oplus 1, N}$, then $\tilde{i}_u\{N(\tilde{s}_u(\gamma^+)) \oplus_1 [\tilde{i}_u(\gamma^-) \otimes_1 \wedge_{i=1}^n (N(\tilde{s}_u(\alpha_i^+)) \oplus_1 (\tilde{i}_u(\alpha_i^-) \otimes_1 \eta_i^-(v)))]\} \leq BKS_n(v) \leq \tilde{i}_u\{N(\tilde{i}_u(\gamma^-)) \oplus_1 [\tilde{s}_u(\gamma^+) \otimes_1 \wedge_{i=1}^n (N(\tilde{i}_u(\alpha_i^-)) \oplus_1 (\tilde{s}_u(\alpha_i^+) \otimes_1 \eta_i^+(v)))]\}$.

On the strength of Theorems 3 and 4, if the operations (i.e., t-norm, t-conorm, and fuzzy negation) are continuous, then the BKS method for multiple rules is stable for the case of (S, N)-implication, or QL-implication, or t-norm implication with $\otimes$. Moreover, if the R-implication and t-norm are continuous, then the BKS method for multiple rules is stable.

Assume that there is a perturbation sequence $([\alpha_{im}^-, \alpha_{im}^+], [\eta_{im}^-, \eta_{im}^+], [\gamma_{im}^-, \gamma_{im}^+])$ of the input $(A_1, \ldots, A_n, B_1, \ldots, B_n, C)$ for the BKS method making $\lim_{m\to\infty} \sup_{u\in U}(\alpha_{im}^+(u) - \alpha_{im}^-(u)) = \lim_{m\to\infty} \sup_{v\in V}(\eta_{im}^+(v) - \eta_{im}^-(v)) = \lim_{m\to\infty} \sup_{u\in U}(\gamma_{im}^+(u) - \gamma_{im}^-(u)) = 0$ hold ($i = 1, 2, \ldots, n$). In Theorems 3 and 4, we analogously make use of λ_m^- and λ_m^+ to signify the homologous lower and upper bounds of the BKS output. When the continuous condition is effective, we gain from Theorems 3 and 4 that $\lim_{m\to\infty} \sup_{v\in V}(\lambda_m^+(v) - \lambda_m^-(v)) = 0$. That is, the output of BKS method for multiple rules converges to a value with meeting the continuity. On the grounds of Definition 15, the BKS algorithm for the situation of multiple rules is stable from the viewpoint of interval perturbation.

IV. Interval Perturbation for BKS Chain Inference

A meaningful generalization of BKS is a chain of fuzzy reasoning. Let $U_1, U_2, \ldots, U_{n+1}$ be $n+1$ universes. $C, A_1 \in F(U_1)$, $B_1, D_1, A_2 \in F(U_2), \ldots,$ and finally $B_n, D_n \in F(U_{n+1})$. We probe into the following reasoning chain:

Input:	C
Rule 1:	A_1 implies B_1
Output 1:	D_1
Rule 2:	A_2 implies B_2
Output 2:	D_2
......	
Output n-1:	D_{n-1}
Rule n:	A_n implies B_n
Final output:	D_n.

Theorem 5: Assume that $A_i \in [\alpha_i^-, \alpha_i^+]$, $B_i \in [\eta_i^-, \eta_i^+]$, and $C \in [\gamma_1^-, \gamma_1^+]$ ($i = 1, 2, \ldots, n$), and that $\otimes$ is a t-norm. The BKS-T method is adopted.

1) If $\rightarrow$ is an R-implication, then $\tilde{s}_{u_n}(\gamma_n^+) \lhd_n (\tilde{i}_{u_n}(\alpha_n^-) \otimes \eta_n^-(u_{n+1})) \leq D_n(u_{n+1}) \leq \tilde{i}_{u_n}(\gamma_n^-) \lhd_n (\tilde{s}_{u_n}(\alpha_n^+) \otimes \eta_n^+(u_{n+1}))$, in which $\gamma_{i+1}^+(u_{i+1}) \triangleq \tilde{i}_{u_i}(\gamma_i^-) \lhd_i (\tilde{s}_{u_i}(\alpha_i^+) \otimes \eta_i^+(u_{i+1}))$, $\gamma_{i+1}^-(u_{i+1}) \triangleq \tilde{s}_{u_i}(\gamma_i^+) \lhd_i (\tilde{i}_{u_i}(\alpha_i^-) \otimes \eta_i^-(u_{i+1}))$ ($i = 1, 2, \ldots, n-1$).
2) If $\rightarrow$ is a t-norm implication, then $\tilde{i}_{u_n}(\gamma_n^-) \lhd_n (\tilde{i}_{u_n}(\alpha_n^-) \otimes \eta_n^-(u_{n+1})) \leq D_n(u_{n+1}) \leq \tilde{s}_{u_n}(\gamma_n^+) \lhd_n (\tilde{s}_{u_n}(\alpha_n^+) \otimes \eta_n^+(u_{n+1}))$, in which $\gamma_{i+1}^+(u_{i+1}) \triangleq \tilde{s}_{u_i}(\gamma_i^+) \lhd_i (\tilde{s}_{u_i}(\alpha_i^+) \otimes \eta_i^+(u_{i+1}))$, $\gamma_{i+1}^-(u_{i+1}) \triangleq \tilde{i}_{u_i}(\gamma_i^-) \lhd_i (\tilde{i}_{u_i}(\alpha_i^-) \otimes \eta_i^-(u_{i+1}))$ ($i = 1, 2, \ldots, n-1$).
3) If $\rightarrow$ is an (S, N)-implication $I_{\oplus 1, N}$, then $\tilde{i}_{u_n}\{N(\tilde{s}_{u_n}(\gamma_n^+)) \oplus_1 (\tilde{i}_{u_n}(\alpha_n^-) \otimes \eta_n^-(u_{n+1}))\} \leq D_n(u_{n+1}) \leq \tilde{i}_{u_n}\{N(\tilde{i}_{u_n}(\gamma_n^-)) \oplus_1 (\tilde{s}_{u_n}(\alpha_n^+) \otimes \eta_n^+(u_{n+1}))\}$, in which $\gamma_{i+1}^+(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{i}_{u_i}(\gamma_i^-)) \oplus_1 (\tilde{s}_{u_i}(\alpha_i^+) \otimes \eta_i^+(u_{i+1}))\}$, $\gamma_{i+1}^-(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{s}_{u_i}(\gamma_i^+)) \oplus_1 (\tilde{i}_{u_i}(\alpha_i^-) \otimes \eta_i^-(u_{i+1}))\}$ ($i = 1, 2, \ldots, n-1$).
4) If $\rightarrow$ is a QL-implication $I_{\otimes 1, \oplus 1, N}$, then $\tilde{i}_{u_n}\{N(\tilde{s}_{u_n}(\gamma_n^+)) \oplus_1 [\tilde{i}_{u_n}(\gamma_n^-) \otimes_1 (\tilde{i}_{u_n}(\alpha_n^-) \otimes \eta_n^-(u_{n+1}))]\} \leq D_n(u_{n+1}) \leq \tilde{i}_{u_n}\{N(\tilde{i}_{u_n}(\gamma_n^-)) \oplus_1 [\tilde{s}_{u_n}(\gamma_n^+) \otimes_1 (\tilde{s}_{u_n}(\alpha_n^+) \otimes \eta_n^+(u_{n+1}))]\}$, in which $\gamma_{i+1}^+(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{i}_{u_i}(\gamma_i^-)) \oplus_1 [\tilde{s}_{u_i}(\gamma_i^+) \otimes_1 (\tilde{s}_{u_i}(\alpha_i^+) \otimes \eta_i^+(u_{i+1}))]\}$, $\gamma_{i+1}^-(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{s}_{u_i}(\gamma_i^+)) \oplus_1 [\tilde{i}_{u_i}(\gamma_i^-) \otimes_1 (\tilde{i}_{u_i}(\alpha_i^-) \otimes \eta_i^-(u_{i+1}))]\}$ ($i = 1, 2, \ldots, n-1$).

Proof: Here, we merely validate 1). It is analogous to 1) of Theorem 1 that $\tilde{s}_{u_1}(\gamma_1^+) \lhd_1 (\tilde{i}_{u_1}(\alpha_1^-) \otimes \eta_1^-(u_2)) \leq D_1(u_2) = BKS(A_1, B_1, C)(u_2) \leq \tilde{i}_{u_1}(\gamma_1^-) \lhd_1 (\tilde{s}_{u_1}(\alpha_1^+) \otimes \eta_1^+(u_2))$.

We denote $\gamma_2^+(u_2) \triangleq \tilde{i}_{u_1}(\gamma_1^-) \lhd_1 (\tilde{s}_{u_1}(\alpha_1^+) \otimes \eta_1^+(u_2)) = \inf_{u_1\in U_1}\{\inf_{u_1\in U_1}(\gamma_1^-(u_1)) \rightarrow (\sup_{u_1\in U_1}(\alpha_1^+(u_1)) \otimes \eta_1^+(u_2))\}$, and $\gamma_2^-(u_2) \triangleq \tilde{s}_{u_1}(\gamma_1^+) \lhd_1 (\tilde{i}_{u_1}(\alpha_1^-) \otimes \eta_1^-(u_2)) = \inf_{u_1\in U_1}\{\sup_{u_1\in U_1}(\gamma_1^+(u_1)) \rightarrow (\inf_{u_1\in U_1}(\alpha_1^-(u_1)) \otimes \eta_1^-(u_2))\}$. Then, we obtain $\tilde{s}_{u_2}(\gamma_2^+) \lhd_2 (\tilde{i}_{u_2}(\alpha_2^-) \otimes \eta_2^-(u_3)) \leq D_2(u_3) = BKS(A_2, B_2, D_1)(u_3) \leq \tilde{i}_{u_2}(\gamma_2^-) \lhd_2 (\tilde{s}_{u_2}(\alpha_2^+) \otimes \eta_2^+(u_3))$.

More generally, we can acquire $\tilde{s}_{u_{i+1}}(\gamma_{i+1}^+) \lhd_{i+1} (\tilde{i}_{u_{i+1}}(\alpha_{i+1}^-) \otimes \eta_{i+1}^-(u_{i+2})) \leq D_{i+1}(u_{i+2}) = BKS(A_{i+1}, B_{i+1}, D_i)(u_{i+2}) \leq \tilde{i}_{u_{i+1}}(\gamma_{i+1}^-) \lhd_{i+1} (\tilde{s}_{u_{i+1}}(\alpha_{i+1}^+) \otimes \eta_{i+1}^+(u_{i+2}))$ ($i =$

$1, 2, \ldots, n-1$), in which $\gamma_{i+1}^{+}(u_{i+1}) \triangleq \tilde{i}_{u_i}(\gamma_i^{-}) \lhd_i (\tilde{s}_{u_i}(\alpha_i^{+}) \otimes \eta_i^{+}(u_{i+1})) = \inf_{u_i \in U_i}\{\inf_{u_i \in U_i}(\gamma_i^{-}(u_i)) \to (\sup_{u_i \in U_i}(\alpha_i^{+}(u_i)) \otimes \eta_i^{+}(u_{i+1}))\}$, and $\gamma_{i+1}^{-}(u_{i+1}) \triangleq \tilde{s}_{u_i}(\gamma_i^{+}) \lhd_i (\tilde{i}_{u_i}(\alpha_i^{-}) \otimes \eta_i^{-}(u_{i+1})) = \inf_{u_i \in U_i}\{\sup_{u_i \in U_i}(\gamma_i^{+}(u_i)) \to (\inf_{u_i \in U_i}(\alpha_i^{-}(u_i)) \otimes \eta_i^{-}(u_{i+1}))\}$.

Finally, we gain $\tilde{s}_{u_n}(\gamma_n^{+}) \lhd_n (\tilde{i}_{u_n}(\alpha_n^{-}) \otimes \eta_n^{-}(u_{n+1})) \leq D_n(u_{n+1}) \leq \tilde{i}_{u_n}(\gamma_n^{-}) \lhd_n (\tilde{s}_{u_n}(\alpha_n^{+}) \otimes \eta_n^{+}(u_{n+1}))$. ■

Theorem 6 is proved in a similar way.

Theorem 6: Assume that $A_i \in [\alpha_i^{-}, \alpha_i^{+}]$, $B_i \in [\eta_i^{-}, \eta_i^{+}]$, and $C \in [\gamma_1^{-}, \gamma_1^{+}]$ ($i = 1, 2, \ldots, n$). The BKS-I method is utilized.

1) If $\to$ is an R-implication, then $\tilde{s}_{u_n}(\gamma_n^{+}) \lhd_n (\tilde{s}_{u_n}(\alpha_n^{+}) \to \eta_n^{-}(u_{n+1})) \leq D_n(u_{n+1}) \leq \tilde{i}_{u_n}(\gamma_n^{-}) \lhd_n (\tilde{i}_{u_n}(\alpha_n^{-}) \to \eta_n^{+}(u_{n+1}))$, in which $\gamma_{i+1}^{+}(u_{i+1}) \triangleq \tilde{i}_{u_i}(\gamma_i^{-}) \lhd_i (\tilde{i}_{u_i}(\alpha_i^{-}) \to \eta_i^{+}(u_{i+1}))$ and $\gamma_{i+1}^{-}(u_{i+1}) \triangleq \tilde{s}_{u_i}(\gamma_i^{+}) \lhd_i (\tilde{s}_{u_i}(\alpha_i^{+}) \to \eta_i^{-}(u_{i+1}))$ ($i = 1, 2, \ldots, n-1$).
2) If $\to$ is a t-norm implication, then $\tilde{i}_{u_n}(\gamma_n^{-}) \lhd_n (\tilde{i}_{u_n}(\alpha_n^{-}) \to \eta_n^{-}(u_{n+1})) \leq D_n(u_{n+1}) \leq \tilde{s}_{u_n}(\gamma_n^{+}) \lhd_n (\tilde{s}_{u_n}(\alpha_n^{+}) \to \eta_n^{+}(u_{n+1}))$, in which $\gamma_{i+1}^{+}(u_{i+1}) \triangleq \tilde{s}_{u_i}(\gamma_i^{+}) \lhd_i (\tilde{s}_{u_i}(\alpha_i^{+}) \to \eta_i^{+}(u_{i+1}))$ and $\gamma_{i+1}^{-}(u_{i+1}) \triangleq \tilde{i}_{u_i}(\gamma_i^{-}) \lhd_i (\tilde{i}_{u_i}(\alpha_i^{-}) \to \eta_i^{-}(u_{i+1}))$ ($i = 1, 2, \ldots, n-1$).
3) If $\to$ is an (S, N)-implication $I_{\oplus_1, N}$, then $\tilde{i}_{u_n}\{N(\tilde{s}_{u_n}(\gamma_n^{+})) \oplus_1 (N(\tilde{s}_{u_n}(\alpha_n^{+})) \oplus_1 \eta_n^{-}(u_{n+1}))\} \leq D_n(u_{n+1}) \leq \tilde{i}_{u_n}\{N(\tilde{i}_{u_n}(\gamma_n^{-})) \oplus_1 (N(\tilde{i}_{u_n}(\alpha_n^{-})) \oplus_1 \eta_n^{+}(u_{n+1}))\}$, in which $\gamma_{i+1}^{+}(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{i}_{u_i}(\gamma_i^{-})) \oplus_1 (N(\tilde{i}_{u_i}(\alpha_i^{-})) \oplus_1 \eta_i^{+}(u_{i+1}))\}$, $\gamma_{i+1}^{-}(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{s}_{u_i}(\gamma_i^{+})) \oplus_1 (N(\tilde{s}_{u_i}(\alpha_i^{+})) \oplus_1 \eta_i^{-}(u_{i+1}))\}$ ($i = 1, 2, \ldots, n-1$).
4) If $\to$ is an QL-implication $I_{\otimes_1, \oplus_1, N}$, then $\tilde{i}_{u_n}\{N(\tilde{s}_{u_n}(\gamma_n^{+})) \oplus_1 [\tilde{i}_{u_n}(\gamma_n^{-}) \otimes_1 (N(\tilde{s}_{u_n}(\alpha_n^{+})) \oplus_1 (\tilde{i}_{u_n}(\alpha_n^{-}) \otimes_1 \eta_n^{-}(u_{n+1})))]\} \leq D_n(u_{n+1}) \leq \tilde{i}_{u_n}\{N(\tilde{i}_{u_n}(\gamma_n^{-})) \oplus_1 [\tilde{s}_{u_n}(\gamma_n^{+}) \otimes_1 (N(\tilde{i}_{u_n}(\alpha_n^{-})) \oplus_1 (\tilde{s}_{u_n}(\alpha_n^{+}) \otimes_1 \eta_n^{+}(u_{n+1})))]\}$, in which $\gamma_{i+1}^{+}(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{i}_{u_i}(\gamma_i^{-})) \oplus_1 [\tilde{s}_{u_i}(\gamma_i^{+}) \otimes_1 (N(\tilde{i}_{u_i}(\alpha_i^{-})) \oplus_1 (\tilde{s}_{u_i}(\alpha_i^{+}) \otimes_1 \eta_i^{+}(u_{i+1})))]\}$, $\gamma_{i+1}^{-}(u_{i+1}) \triangleq \tilde{i}_{u_i}\{N(\tilde{s}_{u_i}(\gamma_i^{+})) \oplus_1 [\tilde{i}_{u_i}(\gamma_i^{-}) \otimes_1 (N(\tilde{s}_{u_i}(\alpha_i^{+})) \oplus_1 (\tilde{i}_{u_i}(\alpha_i^{-}) \otimes_1 \eta_i^{-}(u_{i+1})))]\}$ ($i = 1, 2, \ldots, n-1$).

In accordance with Theorems 5 and 6, if the continuous condition holds for the related operations (i.e., t-norm, t-conorm, and fuzzy negation), then the fuzzy chain reasoning strategy of BKS is stable, in which $\otimes$ and an (S, N)-implication, or a QL-implication, or a t-norm implication are utilized. In addition, if the R-implication and t-norm are continuous, then the BKS chain reasoning is stable for interval perturbation.

Suppose that there is a perturbation sequence $([\alpha_{im}^{-}, \alpha_{im}^{+}], [\eta_{im}^{-}, \eta_{im}^{+}], [\gamma_{1m}^{-}, \gamma_{1m}^{+}])$ of the input $(A_1, \ldots, A_n, B_1, \ldots, B_n, C)$ for BKS chain reasoning letting $\lim_{m\to\infty}\sup_{u\in U}(\alpha_{im}^{+}(u) - \alpha_{im}^{-}(u)) = \lim_{m\to\infty}\sup_{v\in V}(\eta_{im}^{+}(v) - \eta_{im}^{-}(v)) = \lim_{m\to\infty}\sup_{u\in U}(\gamma_{1m}^{+}(u) - \gamma_{1m}^{-}(u)) = 0$ hold ($i = 1, 2, \ldots, n$). In Theorems 5 and 6, we employ $\lambda_m^{-}, \lambda_m^{+}$ to represent the corresponding lower and upper bounds of BKS chain output, viz., $D_n(u_{n+1}) \in [\lambda_m^{-}, \lambda_m^{+}]$ for $([\alpha_{im}^{-}, \alpha_{im}^{+}], [\eta_{im}^{-}, \eta_{im}^{+}], [\gamma_{1m}^{-}, \gamma_{1m}^{+}])$ ($m = 1, 2, \ldots$). When the continuous condition holds, Theorems 5 and 6 imply $\lim_{m\to\infty}\sup_{u_{n+1}\in U_{n+1}}(\lambda_m^{+}(u_{n+1}) - \lambda_m^{-}(u_{n+1})) = 0$. That is, the output of BKS chain reasoning method is converged gradually to a value when the continuous condition is effective. From Definition 15, the BKS chain reasoning method is stable from the perspective of interval perturbation.

V. Simple Perturbations for BKS

A. Simple Perturbations for BKS With One Rule

Here, ten commonly used fuzzy implications are studied.

1) $I_1(p, q) = (1 - p) \vee q$ (the Kleene-Dienes implication, as an (S, N)-implication and QL-implication).
2) $I_2(p, q) = 1 - p + pq$ (the Reichenbach implication, as an (S, N)-implication and QL-implication).
3) $I_3(p, q) = 1 \wedge (1 - p + q)$ (the Lukasiewicz implication, as an R-implication, (S, N)-implication and QL-implication).
4) $I_4(p, q) = (1 - p) \vee (p \wedge q)$ (the Zadeh implication, as an QL-implication).
5) $I_5(p, q) = (p \wedge q) \vee ((1-p) \wedge q) \vee ((1-p) \wedge (1-q))$ ([35]).
6) $I_6(p, q) = (1 - p) \vee (p + q - 1)$ (as an QL-implication [32]).
7) $I_7(p, q) = (1 - p^2) \vee q$ (as an (S, N)-implication [36]).
8) $I_8(p, q) = 1 - p + p^2 q$ (as an QL-implication [32]).
9) $I_9(p, q) = p \wedge q$ (the Mamdani implication, as a t-norm implication).
10) $I_{10}(p, q) = p \times q$ (the Larsen implication, as a t-norm implication).

Lemma 3 [29]: Assume that $A \in F(U), B \in F(V)$, and that β_1 and β_2 are factors of perturbation of A and B. Denote $\Delta I(u, v) = [(A(u)+\beta_1(u)) \to (B(v)+\beta_2(v))] - [A(u) \to B(v)]$. Then, one has the following outcomes.

1) If $\to$ employs I_1, then $(-\beta_1(u)) \wedge \beta_2(v) \leq \Delta I(u, v) \leq (-\beta_1(u)) \vee \beta_2(v)$.
2) If $\to$ employs I_2, then $(-1) \vee (0 \wedge \beta_2(v) - 0 \vee \beta_1(u)) \leq \Delta I(u, v) \leq 1 \wedge (0 \vee \beta_2(v) - 0 \wedge \beta_1(u))$.
3) If $\to$ employs I_3, then $(-1) \vee (0 \wedge (\beta_2(v) - \beta_1(u))) \leq \Delta I(u, v) \leq 1 \wedge (0 \vee (\beta_2(v) - \beta_1(u)))$.
4) If $\to$ employs I_4, then $(-\beta_1(u)) \wedge \beta_1(u) \wedge \beta_2(v) \leq \Delta I(u, v) \leq (-\beta_1(u)) \vee \beta_1(u) \vee \beta_2(v)$.
5) If $\to$ employs I_5, then $\beta_1(u) \wedge \beta_2(v) \wedge (-\beta_1(u) \vee \beta_2(v)) \leq \Delta I(u, v) \leq \beta_1(u) \vee \beta_2(v) \vee (-\beta_1(u) \wedge \beta_2(v))$.
6) If $\to$ employs I_9, then $\beta_1(u) \wedge \beta_2(v) \leq \Delta I(u, v) \leq \beta_1(u) \vee \beta_2(v)$.

We denote $\Delta(v) = BKS(A^*, B^*, C^*)(v) - BKS(A, B, C)(v)$. Thereinto, $A^*(u) = A(u) + \beta_1(u)$, $B^*(v) = B(v) + \beta_2(v)$, and $C^*(u) = C(u) + \beta_3(u)$, in which A^* and C^* are respectively, the perturbations of A and C on U while B^* is the perturbation of B on V.

In the beginning, we investigate the situation of the BKS-I algorithm whose solution is indicated by (9). It is effortless to give evidence of Lemma 4.

Lemma 4: Suppose that $A \in F(U)$ and $B \in F(V)$, and that β_1 and β_2 are the factors of perturbation of A, B in turn. Then, following results hold.

1) If $\to$ employs I_6, then $(-1) \vee [(-\beta_1(u)) \wedge (\beta_1(u) + \beta_2(v))] \leq \Delta I(u, v) \leq 1 \wedge [(-\beta_1(u)) \vee (\beta_1(u) + \beta_2(v))]$.
2) If $\to$ employs I_7, then $(-1) \vee [(-2A(u)\beta_1(u) - (\beta_1(u))^2) \wedge \beta_2(v)] \leq \Delta I(u, v) \leq 1 \wedge [(-2A(u)\beta_1(u) - (\beta_1(u))^2) \vee \beta_2(v)]$.

TABLE I
ESTIMATION OF SIMPLE PERTURBATIONS FOR SINGLE RULE UNDER BKS-I (FOR THEOREM 7)

$\to$	Lower bound of $\Delta(v)$	Upper bound of $\Delta(v)$
I_1	$(-\tilde{s}_u(\beta_1)) \wedge \beta_2(v) \wedge (-\tilde{s}_u(\beta_3))$	$(-\tilde{i}_u(\beta_1)) \vee \beta_2(v) \vee (-\tilde{i}_u(\beta_3))$
I_2	$(-1) \vee [0 \wedge \beta_2(v) - (0 \vee \tilde{s}_u(\beta_1)) - (0 \vee \tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee \beta_2(v) - (0 \wedge \tilde{i}_u(\beta_1)) - (0 \wedge \tilde{i}_u(\beta_3))]$
I_3	$(-1) \vee [0 \wedge (\beta_2(v) - \tilde{s}_u(\beta_1)) + 0 \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee (\beta_2(v) - \tilde{i}_u(\beta_1)) + 0 \vee (-\tilde{i}_u(\beta_3))]$
I_4	$(-\tilde{s}_u(\beta_1)) \wedge \tilde{i}_u(\beta_1) \wedge \beta_2(v) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)$	$(-\tilde{i}_u(\beta_1)) \vee \tilde{s}_u(\beta_1) \vee \beta_2(v) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)$
I_5	$(-\tilde{s}_u(\beta_1)) \wedge \tilde{i}_u(\beta_1) \wedge (-\beta_2(v)) \wedge \beta_2(v) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)$	$(-\tilde{i}_u(\beta_1)) \vee \tilde{s}_u(\beta_1) \vee (-\beta_2(v)) \vee \beta_2(v) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)$
I_6	$(-1) \vee [(-\tilde{s}_u(\beta_3)) \wedge (\tilde{i}_u(\beta_3) + (-\tilde{s}_u(\beta_1)) \wedge (\tilde{i}_u(\beta_1) + \beta_2(v)))]$	$1 \wedge [(-\tilde{i}_u(\beta_3)) \vee (\tilde{s}_u(\beta_3) + (-\tilde{i}_u(\beta_1)) \vee (\tilde{s}_u(\beta_1) + \beta_2(v)))]$
I_7	$(-1) \vee [(-0 \vee 2\tilde{s}_u(\beta_1) - \tilde{s}_u(\beta_1^2)) \wedge \beta_2(v) \wedge (-0 \vee 2\tilde{s}_u(\beta_3) - \tilde{s}_u(\beta_3^2))]$	$1 \wedge [(-0 \wedge 2\tilde{i}_u(\beta_1) - \tilde{i}_u(\beta_1^2)) \vee \beta_2(v) \vee (0 \wedge 2\tilde{i}_u(\beta_3) - \tilde{i}_u(\beta_3^2))]$
I_8	$(-1) \vee [0 \wedge \tilde{i}_u(\beta_1) \wedge (-\tilde{s}_u(\beta_1)) + 0 \wedge \beta_2(v) + 0 \wedge \tilde{i}_u(\beta_3) \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee \tilde{s}_u(\beta_1) \vee (-\tilde{i}_u(\beta_1)) + 0 \vee \beta_2(v) + 0 \vee \tilde{s}_u(\beta_3) \vee (-\tilde{i}_u(\beta_3))]$
I_9	$\tilde{i}_u(\beta_1) \wedge \beta_2(v) \wedge \tilde{i}_u(\beta_3)$	$\tilde{s}_u(\beta_1) \vee \beta_2(v) \vee \tilde{s}_u(\beta_3)$
I_{10}	$(-1) \vee [0 \wedge \tilde{i}_u(\beta_1) + 0 \wedge \beta_2(v) + 0 \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [0 \vee \tilde{s}_u(\beta_1) + 0 \vee \beta_2(v) + 0 \vee \tilde{s}_u(\beta_3)]$

3) If $\to$ employs I_8, then $\Delta I(u,v) = (A(u)B(v) + A^*(u)B(v) - 1) \times \beta_1(u) + (A^*(u))^2\beta_2(v)$, and $(-1) \vee [0 \wedge \beta_1(u) \wedge (-\beta_1(u)) + 0 \wedge \beta_2(v)] \le \Delta I(u,v) \le 1 \wedge [0 \vee \beta_1(u) \vee (-\beta_1(u)) + 0 \vee \beta_2(v)]$.
4) If $\to$ employs I_{10}, then $\Delta I(u,v) = B^*(v)\beta_1(u) + A(u)\beta_2(v)$, and $(-1) \vee [0 \wedge \beta_1(u) + 0 \wedge \beta_2(v)] \le \Delta I(u,v) \le 1 \wedge [0 \vee \beta_1(u) + 0 \vee \beta_2(v)]$.

We denote $\delta(u,v) = [C(u) + \beta_3(u)] \to [(A(u) + \beta_1(u)) \to (B(v) + \beta_2(v))] - [C(u) \to (A(u) \to B(v))]$.

Theorem 7: Let $A, C \in F(U)$ and $B \in F(V)$, and β_1, β_2, and β_3 be the factors of perturbation of A, B, and C in turn. If BKS-I is adopted, then one has the outcomes indicated in Table I.

Proof: At present, we exhibit the proof for the situation of I_8, the others can be gained in an analogous manner. We have from Lemma 4 that

$$\begin{aligned}\delta(u,v) &= [C(u)+\beta_3(u)] \to [(A(u)+\beta_1(u)) \to (B(v)+\beta_2(v))] \\ &\quad - [C(u) \to (A(u) \to B(v))] \\ &\quad = 1 - [C(u)+\beta_3(u)] + [C(u)+\beta_3(u)]^2 \\ &\quad \times \Big[1 - (A(u)+\beta_1(u)) + (A(u)+\beta_1(u))^2 \\ &\qquad \times (B(v)+\beta_2(v))\Big] \\ &\quad - \Big[1 - C(u) + (C(u))^2 \times \Big(1 - A(u) + (A(u))^2 B(v)\Big)\Big] \\ &= (-\beta_3(u)) + \Big[(C(u))^2 + 2C(u)\beta_3(u) + (\beta_3(u))^2\Big] \\ &\quad \times [1 - (A(u)+\beta_1(u)) \\ &\qquad + \Big((A(u))^2 + (\beta_1(u))^2 + 2A(u)\beta_1(u)\Big) \\ &\qquad \times (B(v)+\beta_2(v))] \\ &\quad - (C(u))^2 \times \Big(1 - A(u) + (A(u))^2 B(v)\Big) \\ &= (C(u))^2 \times \Delta I(u,v) + \Big[2C(u)\beta_3(u) + (\beta_3(u))^2\Big] \\ &\quad \times \Big(A^*(u) \to B^*(v)\Big) - \beta_3(u) \\ &= (C(u))^2 \times \Big[\Big(A(u)B(v) + A^*(u)B(v) - 1\Big) \times \beta_1(u) \\ &\qquad + \Big(A^*(u)\Big)^2 \beta_2(v)\Big] \\ &\quad + [C(u)\beta_3(u) + (C(u)+\beta_3(u)) \times \beta_3(u)] \\ &\quad \times \Big(A^*(u) \to B^*(v)\Big) - \beta_3(u) \\ &= (C(u))^2 \times \Big[A(u)B(v) + A^*(u)B(v) - 1\Big] \times \beta_1(u) \\ &\quad + \Big(A^*(u)\Big)^2 (C(u))^2 \beta_2(v) \\ &\quad + \Big[C(u) \times \Big(A^*(u) \to B^*(v)\Big) \\ &\qquad + C^*(u) \times \Big(A^*(u) \to B^*(v)\Big) - 1\Big] \times \beta_3(u).\end{aligned}$$

On the strength of Lemmas 1 and 2, we acquire

$$\begin{aligned}\Delta(v) &= \inf_{u \in U}\{[C(u)+\beta_3(u)] \\ &\qquad \to [(A(u)+\beta_1(u)) \to (B(v)+\beta_2(v))]\} \\ &\quad - \inf_{u \in U}\{[C(u) \to (A(u) \to B(v))]\} \\ &\le \sup_{u \in U}\{[C(u)+\beta_3(u)] \\ &\qquad \to [(A(u)+\beta_1(u)) \to (B(v)+\beta_2(v))] \\ &\qquad - [C(u) \to (A(u) \to B(v))]\} \\ &\le \sup_{u \in U}\Big\{(C(u))^2 \times \Big[A(u)B(v) + A^*(u)B(v) - 1\Big] \times \beta_1(u) \\ &\qquad + \Big(A^*(u)\Big)^2 (C(u))^2 \beta_2(v) \\ &\qquad + \Big[C(u) \times \Big(A^*(u) \to B^*(v)\Big) \\ &\qquad\quad + C^*(u) \times \Big(A^*(u) \to B^*(v)\Big) - 1\Big] \times \beta_3(u)\Big\} \\ &\le \sup_{u \in U}\Big\{(C(u))^2 \times \Big[A(u)B(v) + A^*(u)B(v) - 1\Big] \times \beta_1(u)\Big\} \\ &\quad + \sup_{u \in U}\Big\{\Big(A^*(u)\Big)^2 (C(u))^2 \beta_2(v)\Big\} \\ &\quad + \sup_{u \in U}\Big\{\Big[C(u) \times \Big(A^*(u) \to B^*(v)\Big) + C^*(u) \\ &\qquad \times \Big(A^*(u) \to B^*(v)\Big) - 1\Big] \times \beta_3(u)\Big\}.\end{aligned}$$

In this formula, we determine $\sup_{u \in U}\{(C(u))^2 \times [A(u)B(v) + A^*(u)B(v) - 1] \times \beta_1(u)\}$.

1) If $\sup_{u \in U}\{(C(u))^2 \times [A(u)B(v) + A^*(u)B(v) - 1] \times \beta_1(u)\} \le 0$, then the outcome is evident.
2) If $\sup_{u \in U}\{(C(u))^2 \times [A(u)B(v) + A^*(u)B(v) - 1] \times \beta_1(u)\} > 0$, then two situations are considered. On the one hand, if $\sup_{u \in U}\{\beta_1(u)\} > 0$, then $\sup_{u \in U}\{(C(u))^2 \times$

TABLE II
ESTIMATION OF SIMPLE PERTURBATIONS FOR SINGLE RULE UNDER BKS-T WITH $\otimes_M$ (FOR THEOREM 8)

$\rightarrow$	Lower bound of $\Delta(v)$	Upper bound of $\Delta(v)$
I_1	$\tilde{i}_u(\beta_1)\wedge\beta_2(v)\wedge(-\tilde{s}_u(\beta_3))$	$\tilde{s}_u(\beta_1)\vee\beta_2(v)\vee(-\tilde{i}_u(\beta_3))$
I_2	$(-1)\vee[(0\wedge\tilde{i}_u(\beta_1)+0\wedge\tilde{i}_u(\beta_3))\wedge(0\wedge\beta_2(v)+0\wedge\tilde{i}_u(\beta_3)) -\tilde{s}_u(\beta_3)]$	$1\wedge[(0\vee\tilde{s}_u(\beta_1)+0\vee\tilde{s}_u(\beta_3))\vee(0\vee\beta_2(v)+0\vee\tilde{s}_u(\beta_3)) -\tilde{i}_u(\beta_3)]$
I_3	$(-1)\vee[0\wedge((\tilde{i}_u(\beta_1)\wedge\beta_2(v))-\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee((\tilde{s}_u(\beta_1)\vee\beta_2(v))-\tilde{i}_u(\beta_3))]$
I_4	$\tilde{i}_u(\beta_1)\wedge\beta_2(v)\wedge(-\tilde{s}_u(\beta_3))\wedge\tilde{i}_u(\beta_3)$	$\tilde{s}_u(\beta_1)\vee\beta_2(v)\vee(-\tilde{i}_u(\beta_3))\vee\tilde{s}_u(\beta_3)$
I_5	$(-\tilde{s}_u(\beta_1))\wedge\tilde{i}_u(\beta_1)\wedge(-\beta_2(v))\wedge\beta_2(v)\wedge(-\tilde{s}_u(\beta_3))\wedge\tilde{i}_u(\beta_3)$	$(-\tilde{i}_u(\beta_1))\vee\tilde{s}_u(\beta_1)\vee(-\beta_2(v))\vee\beta_2(v)\vee(-\tilde{i}_u(\beta_3))\vee\tilde{s}_u(\beta_3)$
I_6	$(-1)\vee[(-\tilde{s}_u(\beta_3))\wedge(\tilde{i}_u(\beta_3)+\tilde{i}_u(\beta_1)\wedge\beta_2(v))]$	$1\wedge[(-\tilde{i}_u(\beta_3))\vee(\tilde{s}_u(\beta_3)+\tilde{s}_u(\beta_1)\vee\beta_2(v))]$
I_7	$(-1)\vee[\tilde{i}_u(\beta_1)\wedge\beta_2(v)\wedge(-0\vee2\tilde{s}_u(\beta_3)-\tilde{s}_u(\beta_3^2))]$	$1\wedge[\tilde{s}_u(\beta_1)\vee\beta_2(v)\vee(-0\wedge2\tilde{i}_u(\beta_3)-\tilde{i}_u(\beta_3^2))]$
I_8	$(-1)\vee[(0\wedge\tilde{i}_u(\beta_1)+0\wedge\tilde{i}_u(2\beta_3)+0\wedge\tilde{i}_u(\beta_3^2))\wedge (0\wedge\beta_2(v)+0\wedge\tilde{i}_u(2\beta_3)+0\wedge\tilde{i}_u(\beta_3^2))-\tilde{s}_u(\beta_3)]$	$1\wedge[(0\vee\tilde{s}_u(\beta_1)+0\vee\tilde{s}_u(2\beta_3)+0\vee\tilde{s}_u(\beta_3^2))\vee (0\vee\beta_2(v)+0\vee\tilde{s}_u(2\beta_3)+0\vee\tilde{s}_u(\beta_3^2))-\tilde{i}_u(\beta_3)]$
I_9	$\tilde{i}_u(\beta_1)\wedge\beta_2(v)\wedge\tilde{i}_u(\beta_3)$	$\tilde{s}_u(\beta_1)\vee\beta_2(v)\vee\tilde{s}_u(\beta_3)$
I_{10}	$(-1)\vee[0\wedge\tilde{i}_u(\beta_1)+0\wedge\tilde{i}_u(\beta_3)]\wedge[0\wedge\beta_2(v)+0\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[0\vee\tilde{s}_u(\beta_1)+0\vee\tilde{s}_u(\beta_3)]\vee[0\vee\beta_2(v)+0\vee\tilde{s}_u(\beta_3)]$

TABLE III
ESTIMATION OF SIMPLE PERTURBATIONS FOR SINGLE RULE UNDER BKS-T WITH $\otimes_P$ (FOR THEOREM 9)

$\rightarrow$	Lower bound of $\Delta(v)$	Upper bound of $\Delta(v)$
I_1	$(-1)\vee[(0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v))\wedge(-\tilde{s}_u(\beta_3))]$	$1\wedge[(0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v))\vee(-\tilde{i}_u(\beta_3))]$
I_2	$(-1)\vee[0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v)-(0\vee\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v)-(0\wedge\tilde{i}_u(\beta_3))]$
I_3	$(-1)\vee[0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v)+0\wedge(-\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v)+0\vee(-\tilde{i}_u(\beta_3))]$
I_4	$(-1)\vee[(0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v))\wedge(-\tilde{s}_u(\beta_3))\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[(0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v))\vee(-\tilde{i}_u(\beta_3))\vee\tilde{s}_u(\beta_3)]$
I_5	$(-1)\vee[(-0\vee\tilde{s}_u(\beta_1)-0\vee\beta_2(v))\wedge(0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v))\wedge (-\tilde{s}_u(\beta_3))\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[(-0\wedge\tilde{i}_u(\beta_1)-0\wedge\beta_2(v))\vee(0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v))\vee (-\tilde{i}_u(\beta_3))\vee\tilde{s}_u(\beta_3)]$
I_6	$(-1)\vee[(-\tilde{s}_u(\beta_3))\wedge(0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v)+\tilde{i}_u(\beta_3))]$	$1\wedge[(-\tilde{i}_u(\beta_3))\vee(0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v)+\tilde{s}_u(\beta_3))]$
I_7	$(-1)\vee[(0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v))\wedge(-0\vee2\tilde{s}_u(\beta_3)-\tilde{s}_u(\beta_3^2))]$	$1\wedge[(0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v))\vee(-0\wedge2\tilde{i}_u(\beta_3)-\tilde{i}_u(\beta_3^2))]$
I_8	$(-1)\vee[0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v)+0\wedge\tilde{i}_u(\beta_3)\wedge(-\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v)+0\vee\tilde{s}_u(\beta_3)\vee(-\tilde{i}_u(\beta_3))]$
I_9	$(-1)\vee[(0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v))\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[(0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v))\vee\tilde{s}_u(\beta_3)]$
I_{10}	$(-1)\vee[0\wedge\tilde{i}_u(\beta_1)+0\wedge\beta_2(v)+0\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[0\vee\tilde{s}_u(\beta_1)+0\vee\beta_2(v)+0\vee\tilde{s}_u(\beta_3)]$

$[A(u)B(v)+A^*(u)B(v)-1]\times\beta_1(u)\}\leq\sup_{u\in U}\{\beta_1(u)\}=\tilde{s}_u(\beta_1)$. On the other hand, if $\sup_{u\in U}\{\beta_1(u)\}<0$, then $\sup_{u\in U}\{(C(u))^2\times[A(u)B(v)+A^*(u)B(v)-1]\times\beta_1(u)\}\leq\sup_{u\in U}\{(-\beta_1(u))\}=-\inf_{u\in U}\{\beta_1(u)\}=-\tilde{i}_u(\beta_1)$.

Placing these two expressions together, we gain $\sup_{u\in U}\{(C(u))^2\times[A(u)B(v)+A^*(u)B(v)-1]\times\beta_1(u)\}\leq 0\vee\tilde{s}_u(\beta_1)\vee(-\tilde{i}_u(\beta_1))$.

Analogously, we obtain $\sup_{u\in U}\{(A^*(u))^2(C(u))^2\beta_2(v)\}\leq 0\vee\beta_2(v)$, and $\sup_{u\in U}\{[C(u)\times(A^*(u)\rightarrow B^*(v))+C^*(u)\times(A^*(u)\rightarrow B^*(v))-1]\times\beta_3(u)\}\leq 0\vee\tilde{s}_u(\beta_3)\vee(-\tilde{i}_u(\beta_3))$.

As a consequence, we have $\Delta(v)\leq 0\vee\tilde{s}_u(\beta_1)\vee(-\tilde{i}_u(\beta_1))+0\vee\beta_2(v)+0\vee\tilde{s}_u(\beta_3)\vee(-\tilde{i}_u(\beta_3))$.

Distinctly, $\Delta(v)\leq 1$, hence it implies $\Delta(v)\leq 1\wedge[0\vee\tilde{s}_u(\beta_1)\vee(-\tilde{i}_u(\beta_1))+0\vee\beta_2(v)+0\vee\tilde{s}_u(\beta_3)\vee(-\tilde{i}_u(\beta_3))]$.

In an analogous situation, we obtain $\Delta(v)\geq(-1)\vee[0\wedge\tilde{i}_u(\beta_1)\wedge(-\tilde{s}_u(\beta_1))+0\wedge\beta_2(v)+0\wedge\tilde{i}_u(\beta_3)\wedge(-\tilde{s}_u(\beta_3))]$.

Summarizing the above findings, one achieves $(-1)\vee[0\wedge\tilde{i}_u(\beta_1)\wedge(-\tilde{s}_u(\beta_1))+0\wedge\beta_2(v)+0\wedge\tilde{i}_u(\beta_3)\wedge(-\tilde{s}_u(\beta_3))]\leq\Delta(v)\leq 1\wedge[0\vee\tilde{s}_u(\beta_1)\vee(-\tilde{i}_u(\beta_1))+0\vee\beta_2(v)+0\vee\tilde{s}_u(\beta_3)\vee(-\tilde{i}_u(\beta_3))]$. ■

In addition, we reveal the situation of the BKS-T algorithm, that is, the solution is demonstrated by (8). It is easy to gain Lemma 5.

Lemma 5: Suppose that $A\in F(U)$ and $B\in F(V)$, and that β_1 and β_2 are factors of perturbation of A and B in turn. Denote $\Delta T(u,v)=[(A(u)+\beta_1(u))\otimes(B(v)+\beta_2(v))]-[A(u)\otimes B(v)]$. Then, we have the following outcomes.

1) If $\otimes$ is $\otimes_M$, then $\beta_1(u)\wedge\beta_2(v)\leq\Delta T(u,v)\leq\beta_1(u)\vee\beta_2(v)$.
2) If $\otimes$ employs $\otimes_P$, then $\Delta T(u,v)=B^*(v)\beta_1(u)+A(u)\beta_2(v)$, and $(-1)\vee[0\wedge\beta_1(u)+0\wedge\beta_2(v)]\leq\Delta T(u,v)\leq 1\wedge[0\vee\beta_1(u)+0\vee\beta_2(v)]$.
3) If $\otimes$ takes $\otimes_L$, then $(-1)\vee[(\beta_1(u)+\beta_2(v))\wedge 0]\leq\Delta T(u,v)\leq 1\wedge[(\beta_1(u)+\beta_2(v))\vee 0]$.

Theorems 8–10 are validated as the previous ones.

Theorem 8: Suppose that $A,C\in F(U)$ and $B\in F(V)$, and β_1,β_2, and β_2 are the factors of perturbation of A,B, and C in turn. If BKS-T is employed and $\otimes$ is $\otimes_M$, then the results in Table II hold for $I_1,I_2,\ldots,I_{10}$.

Theorem 9: Suppose that $A,C\in F(U)$ and $B\in F(V)$, and that β_1,β_2, and β_2 are the factors of perturbation of A,B, and C in turn. If BKS-T is utilized, and $\otimes$ takes $\otimes_P$, then we obtain the outcomes in Table III for I_1–I_{10}.

Theorem 10: Suppose that $A,C\in F(U)$ and $B\in F(V)$, and that β_1,β_2, and β_2 are the factors of perturbation of A,B,

TABLE IV
ESTIMATION OF SIMPLE PERTURBATIONS FOR SINGLE RULE UNDER BKS-T WITH $\otimes_L$ (FOR THEOREM 10)

$\rightarrow$	Lower bound of $\Delta(v)$	Upper bound of $\Delta(v)$
I_1	$(-1) \vee [0 \wedge (\tilde{i}_u(\beta_1) + \beta_2(v)) \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee (\tilde{s}_u(\beta_1) + \beta_2(v)) \vee (-\tilde{i}_u(\beta_3))]$
I_2	$(-1) \vee [0 \wedge \tilde{i}_u(\beta_1) + 0 \wedge \beta_2(v) - 0 \vee \tilde{s}_u(\beta_3)]$	$1 \wedge [0 \vee \tilde{s}_u(\beta_1) + 0 \vee \beta_2(v) - 0 \wedge \tilde{i}_u(\beta_3)]$
I_3	$(-1) \vee [0 \wedge \tilde{i}_u(\beta_1) + 0 \wedge \beta_2(v) + 0 \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee \tilde{s}_u(\beta_1) + 0 \vee \beta_2(v) + 0 \vee (-\tilde{i}_u(\beta_3))]$
I_4	$(-1) \vee [(0 \wedge (\tilde{i}_u(\beta_1) + \beta_2(v))) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [(0 \vee (\tilde{s}_u(\beta_1) + \beta_2(v))) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)]$
I_5	$(-1) \vee [0 \wedge (-\tilde{s}_u(\beta_1) - \beta_2(v)) \wedge (\tilde{i}_u(\beta_1) + \beta_2(v)) \wedge (-\tilde{s}_u(\beta_3))$ $\wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [0 \vee (-\tilde{i}_u(\beta_1) - \beta_2(v)) \vee (\tilde{s}_u(\beta_1) + \beta_2(v)) \vee (-\tilde{i}_u(\beta_3))$ $\vee \tilde{s}_u(\beta_3)]$
I_6	$(-1) \vee [(-\tilde{s}_u(\beta_3)) \wedge (0 \wedge (\tilde{i}_u(\beta_1) + \beta_2(v)) + \tilde{i}_u(\beta_3))]$	$1 \wedge [(-\tilde{i}_u(\beta_3)) \vee (0 \vee (\tilde{s}_u(\beta_1) + \beta_2(v)) + \tilde{s}_u(\beta_3))]$
I_7	$(-1) \vee [0 \wedge (\tilde{i}_u(\beta_1) + \beta_2(v)) \wedge (-0 \vee 2\tilde{s}_u(\beta_3) - \tilde{s}_u(\beta_3^2))]$	$1 \wedge [0 \vee (\tilde{s}_u(\beta_1) + \beta_2(v)) \vee (-0 \wedge 2\tilde{i}_u(\beta_3) - \tilde{i}_u(\beta_3^2))]$
I_8	$(-1) \vee [0 \wedge \tilde{i}_u(\beta_1) + 0 \wedge \beta_2(v) + 0 \wedge \tilde{i}_u(\beta_3) \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee \tilde{s}_u(\beta_1) + 0 \vee \beta_2(v) + 0 \vee \tilde{s}_u(\beta_3) \vee (-\tilde{i}_u(\beta_3))]$
I_9	$(-1) \vee [0 \wedge (\tilde{i}_u(\beta_1) + \beta_2(v)) \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [0 \vee (\tilde{s}_u(\beta_1) + \beta_2(v)) \vee \tilde{s}_u(\beta_3)]$
I_{10}	$(-1) \vee [0 \wedge \tilde{i}_u(\beta_1) + 0 \wedge \beta_2(v) + 0 \wedge \tilde{i}_u(\beta_3) \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee \tilde{s}_u(\beta_1) + 0 \vee \beta_2(v) + 0 \vee \tilde{s}_u(\beta_3) \vee (-\tilde{i}_u(\beta_3))]$

TABLE V
ESTIMATION OF SIMPLE PERTURBATIONS FOR MULTIPLE RULES UNDER BKS-I (FOR THEOREM 11)

$\rightarrow$	Lower bound of $\Delta_n(v)$	Upper bound of $\Delta_n(v)$
I_1	$[\wedge_{i=1}^n(-\tilde{s}_u(\beta_{1i}))] \wedge [\wedge_{i=1}^n \beta_{2i}(v)] \wedge (-\tilde{s}_u(\beta_3))$	$[\vee_{i=1}^n(-\tilde{i}_u(\beta_{1i}))] \vee [\vee_{i=1}^n \beta_{2i}(v)] \vee (-\tilde{i}_u(\beta_3))$
I_2	$(-1) \vee [0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) - 0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) - 0 \vee \tilde{s}_u(\beta_3)]$	$1 \wedge [0 \vee (\vee_{i=1}^n \beta_{2i}(v)) - 0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) - 0 \wedge \tilde{i}_u(\beta_3)]$
I_3	$(-1) \vee [0 \wedge \wedge_{i=1}^n(\beta_{2i}(v) - \tilde{s}_u(\beta_{1i})) + 0 \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee \vee_{i=1}^n(\beta_{2i}(v) - \tilde{i}_u(\beta_{1i})) + 0 \vee (-\tilde{i}_u(\beta_3))]$
I_4	$[\wedge_{i=1}^n(-\tilde{s}_u(\beta_{1i}))] \wedge [\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})] \wedge [\wedge_{i=1}^n \beta_{2i}(v)] \wedge (-\tilde{s}_u(\beta_3))$ $\wedge \tilde{i}_u(\beta_3)$	$[\vee_{i=1}^n(-\tilde{i}_u(\beta_{1i}))] \vee [\vee_{i=1}^n \tilde{s}_u(\beta_{1i})] \vee [\vee_{i=1}^n \beta_{2i}(v)] \vee (-\tilde{i}_u(\beta_3))$ $\vee \tilde{s}_u(\beta_3)$
I_5	$(\wedge_{i=1}^n(-\tilde{s}_u(\beta_{1i}))) \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n(-\beta_{2i}(v)))\wedge$ $(\wedge_{i=1}^n \beta_{2i}(v)) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)$	$(\vee_{i=1}^n(-\tilde{i}_u(\beta_{1i}))) \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n(-\beta_{2i}(v)))\vee$ $(\vee_{i=1}^n \beta_{2i}(v)) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)$
I_6	$(-1) \vee [(-\tilde{s}_u(\beta_3)) \wedge (\tilde{i}_u(\beta_3) + (\wedge_{i=1}^n(-\tilde{s}_u(\beta_{1i}))\wedge$ $(\wedge_{i=1}^n \tilde{i}_u(\beta_{1i}) + \wedge_{i=1}^n \beta_{2i}(v))))]$	$1 \wedge [(-\tilde{i}_u(\beta_3)) \vee (\tilde{s}_u(\beta_3) + (\vee_{i=1}^n(-\tilde{i}_u(\beta_{1i}))\vee$ $(\vee_{i=1}^n \tilde{s}_u(\beta_{1i}) + \vee_{i=1}^n \beta_{2i}(v))))]$
I_7	$(-1) \vee [(\wedge_{i=1}^n(-0 \vee 2\tilde{s}_u(\beta_{1i}) - \tilde{s}_u(\beta_{1i}^2))) \wedge (\wedge_{i=1}^n \beta_{2i}(v))\wedge$ $(-0 \vee 2\tilde{s}_u(\beta_3) - \tilde{s}_u(\beta_3^2))]$	$1 \wedge [(\vee_{i=1}^n(-0 \wedge 2\tilde{i}_u(\beta_{1i}) - \tilde{i}_u(\beta_{1i}^2))) \vee (\vee_{i=1}^n \beta_{2i}(v))\vee$ $(-0 \wedge 2\tilde{i}_u(\beta_3) - \tilde{i}_u(\beta_3^2))]$
I_8	$(-1) \vee [0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_1)) \wedge (\wedge_{i=1}^n(-\tilde{s}_u(\beta_1)))+$ $0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + 0 \wedge \tilde{i}_u(\beta_3) \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_1)) \vee (\vee_{i=1}^n(-\tilde{i}_u(\beta_1)))+$ $0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + 0 \vee \tilde{s}_u(\beta_3) \vee (-\tilde{i}_u(\beta_3))]$
I_9	$(\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v)) \wedge \tilde{i}_u(\beta_3)$	$(\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n \beta_{2i}(v)) \vee \tilde{s}_u(\beta_3)$
I_{10}	$(-1) \vee [0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + 0 \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + 0 \vee \tilde{s}_u(\beta_3)]$

and C in turn. If BKS-T is used and $\otimes$ takes $\otimes_L$, then the conclusions in Table IV are derived.

Suppose that $A, C \in F(U)$ and $B \in F(V)$. If there is a small positive real number ε, and functions $\theta_k(u) : U \rightarrow [-1, 1]$ for $k = 1, 3$ or $\theta_2(v) : V \rightarrow [-1, 1]$ letting $\beta_k(u) = \varepsilon\theta_k(u)$ hold where $k = 1, 3$ and $\beta_2(v) = \varepsilon\theta_2(v)$, then we acquire from Theorems 7–10 that $\lim_{\varepsilon \rightarrow 0} BKS(A^*, B^*, C^*)(v) = BKS(A, B, C)(v)$. Hence, the BKS algorithm is stable in view of Definition 16-1).

From another perspective, assume that there is a perturbation sequence $\{(A_m^*, B_m^*, C_m^*)\}$ with regard to $\{(\beta_{1m}(u), \beta_{2m}(v), \beta_{3m}(u))\}$, viz., $A_m^*(u) = A(u) + \beta_{1m}(u)$, $B_m^*(v) = B(v) + \beta_{2m}(v)$, and $C_m^*(u) = C(u) + \beta_{3m}(u)$ $(m = 1, 2, \ldots,)$. Let $\lim_{m\rightarrow\infty} \sup_{u\in U} |\beta_{1m}(u)| = \lim_{m\rightarrow\infty} \sup_{v\in V} |\beta_{2m}(v)| = \lim_{m\rightarrow\infty} \sup_{u\in U} |\beta_{3m}(u)| = 0$ holds for $\beta_{1m}(u), \beta_{2m}(v), \beta_{3m}(u)$. Then, from Theorems 7–10, it follows that $\lim_{m\rightarrow\infty} BKS(A_m^*, B_m^*, C_m^*)(v) = BKS(A, B, C)(v)$. In consequence, it follows from Definition 16-2) that the BKS algorithm is asymptotic stable.

All in all, the BKS algorithm for the situation of a single rule is stable from the perspective of simple perturbation.

B. Simple Perturbations for BKS With Multiple Rules

For the situation of multiple rules, we employ the notation $\Delta_n(v) = BKS(A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_n^*, C^*)(v) - BKS(A_1, \ldots, A_n, B_1, \ldots, B_n, C)(v)$. Thereinto, $A_i^*(u) = A_i(u) + \beta_{1i}(u)$, $B_i^*(v) = B_i(v) + \beta_{2i}(v)$, and $C^*(u) = C(u) + \beta_3(u)$, in which A_i^* and C^* are, respectively, the perturbations of A_i, C on U while B_i^* is the perturbation of B_i on V $(i = 1, 2, \ldots, n)$.

First, we analyze the BKS-I algorithm, viz., the expression (7). Denote $\delta_n(u, v) = [(C(u) + \beta_3(u)) \rightarrow \wedge_{i=1}^n((A_i(u) + \beta_{1i}(u)) \rightarrow (B_i(v)+\beta_{2i}(v)))] - [C(u) \rightarrow \wedge_{i=1}^n(A_i(u) \rightarrow B_i(v))]$. Theorem 11 can be verified in a similar fashion.

Theorem 11: Assume that $A_i, C \in F(U)$ and $B_i \in F(V)$, and that β_{1i}, β_{2i}, and β_3 are the factors of perturbation of A_i, B_i, and C in turn $(i = 1, 2, \ldots, n)$. If BKS-I is employed, then the results for I_1–I_{10} are given below in Table V.

TABLE VI
ESTIMATION OF SIMPLE PERTURBATIONS FOR MULTIPLE RULES UNDER BKS-T WITH $\otimes_M$ (FOR THEOREM 12)

$\rightarrow$	Lower bound of $\Delta_n(v)$	Upper bound of $\Delta_n(v)$
I_1	$(\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v)) \wedge (-\tilde{s}_u(\beta_3))$	$(\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n \beta_{2i}(v)) \vee (-\tilde{i}_u(\beta_3))$
I_2	$(-1) \vee [(0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge \tilde{i}_u(\beta_3)) \wedge (0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + 0 \wedge \tilde{i}_u(\beta_3)) - \tilde{s}_u(\beta_3)]$	$1 \wedge [(0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee \tilde{s}_u(\beta_3)) \vee (0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + 0 \vee \tilde{s}_u(\beta_3)) - \tilde{i}_u(\beta_3)]$
I_3	$(-1) \vee [0 \wedge (((\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v))) - \tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee (((\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n \beta_{2i}(v))) - \tilde{i}_u(\beta_3))]$
I_4	$(\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v)) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)$	$(\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n \beta_{2i}(v)) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)$
I_5	$(\wedge_{i=1}^n (-\tilde{s}_u(\beta_{1i}))) \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n (-\beta_{2i}(v))) \wedge (\wedge_{i=1}^n \beta_{2i}(v)) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)$	$(\vee_{i=1}^n (-\tilde{i}_u(\beta_{1i}))) \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n (-\beta_{2i}(v))) \vee (\vee_{i=1}^n \beta_{2i}(v)) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)$
I_6	$(-1) \vee [(-\tilde{s}_u(\beta_3)) \wedge (\tilde{i}_u(\beta_3) + (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v)))]$	$1 \wedge [(-\tilde{i}_u(\beta_3)) \vee (\tilde{s}_u(\beta_3) + (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n \beta_{2i}(v)))]$
I_7	$(-1) \vee [(\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v)) \wedge (-0 \vee 2\tilde{s}_u(\beta_3) - \tilde{s}_u(\beta_3^2))]$	$1 \wedge [(\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n \beta_{2i}(v)) \vee (-0 \wedge 2\tilde{i}_u(\beta_3) - \tilde{i}_u(\beta_3^2))]$
I_8	$(-1) \vee [(0 \wedge \wedge_{i=1}^n \tilde{i}_u(\beta_{1i}) + 0 \wedge \tilde{i}_u(2\beta_3) + 0 \wedge \tilde{i}_u(\beta_3^2)) \wedge (0 \wedge \wedge_{i=1}^n \beta_{2i}(v) + 0 \wedge \tilde{i}_u(2\beta_3) + 0 \wedge \tilde{i}_u(\beta_3^2)) - \tilde{s}_u(\beta_3)]$	$1 \wedge [(0 \vee \vee_{i=1}^n \tilde{s}_u(\beta_{1i}) + 0 \vee \tilde{s}_u(2\beta_3) + 0 \vee \tilde{s}_u(\beta_3^2)) \vee (0 \vee \vee_{i=1}^n \beta_{2i}(v) + 0 \vee \tilde{s}_u(2\beta_3) + 0 \vee \tilde{s}_u(\beta_3^2)) - \tilde{i}_u(\beta_3)]$
I_9	$(\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v)) \wedge \tilde{i}_u(\beta_3)$	$(\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) \vee (\vee_{i=1}^n \beta_{2i}(v)) \vee \tilde{s}_u(\beta_3)$
I_{10}	$(-1) \vee [(0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge \tilde{i}_u(\beta_3)) \wedge (0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + 0 \wedge \tilde{i}_u(\beta_3))$	$1 \wedge [(0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee \tilde{s}_u(\beta_3)) \vee (0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + 0 \vee \tilde{s}_u(\beta_3))]$

TABLE VII
ESTIMATION OF SIMPLE PERTURBATIONS FOR MULTIPLE RULES UNDER BKS-T WITH $\otimes_P$ (FOR THEOREM 13)

$\rightarrow$	Lower bound of $\Delta_n(v)$	Upper bound of $\Delta_n(v)$
I_1	$(-1) \vee [(0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v))) \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [(0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v))) \vee (-\tilde{i}_u(\beta_3))]$
I_2	$(-1) \vee [0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) - (0 \vee \tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v)) - (0 \wedge \tilde{i}_u(\beta_3))]$
I_3	$(-1) \vee [0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + 0 \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + 0 \vee (-\tilde{i}_u(\beta_3))]$
I_4	$(-1) \vee [(0 \wedge \wedge_{i=1}^n \tilde{i}_u(\beta_{1i}) + 0 \wedge \wedge_{i=1}^n \beta_{2i}(v)) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [(0 \vee \vee_{i=1}^n \tilde{s}_u(\beta_{1i}) + 0 \vee \vee_{i=1}^n \beta_{2i}(v)) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)]$
I_5	$(-1) \vee [\wedge_{i=1}^n (-0 \vee \tilde{s}_u(\beta_{1i}) - 0 \vee \beta_{2i}(v)) \wedge (0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v))) \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [\vee_{i=1}^n (-0 \wedge \tilde{i}_u(\beta_{1i}) - 0 \wedge \beta_{2i}(v)) \vee (0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v))) \vee (-\tilde{i}_u(\beta_3)) \vee \tilde{s}_u(\beta_3)]$
I_6	$(-1) \vee [(-\tilde{s}_u(\beta_3)) \wedge (0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + \tilde{i}_u(\beta_3))]$	$1 \wedge [(-\tilde{i}_u(\beta_3)) \vee (0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + \tilde{s}_u(\beta_3))]$
I_7	$(-1) \vee [(0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v))) \wedge (-0 \vee 2\tilde{s}_u(\beta_3) - \tilde{s}_u(\beta_3^2))]$	$1 \wedge [(0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v))) \vee (-0 \wedge 2\tilde{i}_u(\beta_3) - \tilde{i}_u(\beta_3^2))]$
I_8	$(-1) \vee [0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + 0 \wedge \tilde{i}_u(\beta_3) \wedge (-\tilde{s}_u(\beta_3))]$	$1 \wedge [0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + 0 \vee \tilde{s}_u(\beta_3) \vee (-\tilde{i}_u(\beta_3))]$
I_9	$(-1) \vee [(0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v))) \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [(0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v))) \vee \tilde{s}_u(\beta_3)]$
I_{10}	$(-1) \vee [0 \wedge (\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) + 0 \wedge (\wedge_{i=1}^n \beta_{2i}(v)) + 0 \wedge \tilde{i}_u(\beta_3)]$	$1 \wedge [0 \vee (\vee_{i=1}^n \tilde{s}_u(\beta_{1i})) + 0 \vee (\vee_{i=1}^n \beta_{2i}(v)) + 0 \vee \tilde{s}_u(\beta_3)]$

Afterward, we look at the BKS-T algorithm. The corresponding solution comes in the form (6). We utilize the notation $\xi_n(u, v) = \{[C(u) + \beta_3(u)] \rightarrow \vee_{i=1}^n [(A_i(u) + \beta_{1i}(u)) \otimes (B_i(v) + \beta_{2i}(v))]\} - \{C(u) \rightarrow \vee_{i=1}^n [A_i(u) \otimes B_i(v)]\}$.

Theorems 12–14 are proved in the same manner.

Theorem 12: Assume that $A_i, C \in F(U)$ and $B_i \in F(V)$, and that β_{1i}, β_{2i}, and β_3 are the factors of perturbation of A_i, B_i, and C in turn ($i = 1, 2, \ldots, n$). If BKS-T is adopted, and $\otimes$ takes $\otimes_M$, then the conclusions are provided in Table VI.

Theorem 13: Assume that $A_i, C \in F(U)$ and $B_i \in F(V)$, and that β_{1i}, β_{2i}, and β_3 are the factors of perturbation of A_i, B_i, and C in turn ($i = 1, 2, \ldots, n$). If BKS-T is used, and $\otimes$ takes $\otimes_P$, then we have the outcomes in Table VII.

Theorem 14: Let $A_i, C \in F(U), B_i \in F(V)$, and $\beta_{1i}, \beta_{2i}, \beta_3$ be the factors of perturbation of A_i, B_i, C in turn ($i = 1, 2, \ldots, n$). The BKS-T with $\otimes$ takes $\otimes_L$ yields the results in Table VIII.

Let $A_1, \ldots, A_n, C \in F(U)$, and $B_1, \ldots, B_n \in F(V)$. If there exists a small positive real number ε, and functions $\theta_{ki}(u) : U \rightarrow [-1, 1]$ for $k = 1, 3$ or $\theta_{2i}(v) : V \rightarrow [-1, 1]$ such that $\beta_{ki}(u) = \varepsilon\theta_{ki}(u)$ where $k = 1, 3$ and $\beta_{2i}(v) = \varepsilon\theta_{2i}(v)$ ($i = 1, 2, \ldots, n$), then we obtain from Theorems 11–14 that $\lim_{\varepsilon \to 0} BKS(A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_n^*, C^*)(v) = BKS(A_1, \ldots, A_n, B_1, \ldots, B_n, C)(v)$. Consequently, the BKS algorithm is stable on the strength of Definition 16-1).

From a different viewpoint, assume that there is a perturbation sequence $\{(A_{1m}^*, \ldots, A_{nm}^*, B_{1m}^*, \ldots, B_{nm}^*, C_m^*)\}$ with regard to $\{(\beta_{11m}(u), \ldots, \beta_{1nm}(u), \beta_{21m}(v), \ldots, \beta_{2nm}(v), \beta_{3m}(u))\}$, that is, $A_{1m}^*(u) = A_1(u) + \beta_{11m}(u), \ldots, A_{nm}^*(u) = A_n(u) + \beta_{1nm}(u)$, $B_{1m}^*(v) = B_1(v) + \beta_{21m}(v), \ldots, B_{nm}^*(v) = B_n(v) + \beta_{2nm}(v)$, $C_m^*(u) = C(u) + \beta_{3m}(u)$

TABLE VIII
ESTIMATION OF SIMPLE PERTURBATIONS FOR MULTIPLE RULES UNDER BKS-T WITH $\otimes_L$ (FOR THEOREM 14)

$\to$	Lower bound of $\Delta_n(v)$	Upper bound of $\Delta_n(v)$
I_1	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i})+\wedge_{i=1}^n\beta_{2i}(v))\wedge(-\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i})+\vee_{i=1}^n\beta_{2i}(v))\vee(-\tilde{i}_u(\beta_3))]$
I_2	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i}))+0\wedge(\wedge_{i=1}^n\beta_{2i}(v))-0\vee\tilde{s}_u(\beta_3)]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i}))+0\vee(\vee_{i=1}^n\beta_{2i}(v))-0\wedge\tilde{i}_u(\beta_3)]$
I_3	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i}))+0\wedge(\wedge_{i=1}^n\beta_{2i}(v))+0\wedge(-\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i}))+0\vee(\vee_{i=1}^n\beta_{2i}(v))+0\vee(-\tilde{i}_u(\beta_3))]$
I_4	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i})+\wedge_{i=1}^n\beta_{2i}(v))\wedge(-\tilde{s}_u(\beta_3))\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i})+\vee_{i=1}^n\beta_{2i}(v))\vee(-\tilde{i}_u(\beta_3))\vee\tilde{s}_u(\beta_3)]$
I_5	$(-1)\vee[0\wedge(\wedge_{i=1}^n(-\tilde{s}_u(\beta_{1i})-\beta_{2i}(v)))\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i})+$ $\wedge_{i=1}^n\beta_{2i}(v))\wedge(-\tilde{s}_u(\beta_3))\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[0\vee(\vee_{i=1}^n(-\tilde{i}_u(\beta_{1i})-\beta_{2i}(v)))\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i})+$ $\vee_{i=1}^n\beta_{2i}(v))\vee(-\tilde{i}_u(\beta_3))\vee\tilde{s}_u(\beta_3)]$
I_6	$(-1)\vee[(-\tilde{s}_u(\beta_3))\wedge(0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i})+\wedge_{i=1}^n\beta_{2i}(v))+\tilde{i}_u(\beta_3))]$	$1\wedge[(-\tilde{i}_u(\beta_3))\vee(0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i})+\vee_{i=1}^n\beta_{2i}(v))+\tilde{s}_u(\beta_3))]$
I_7	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i})+\wedge_{i=1}^n\beta_{2i}(v))\wedge(-0\vee2\tilde{s}_u(\beta_3)-$ $\tilde{s}_u(\beta_3^2))]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i})+\vee_{i=1}^n\beta_{2i}(v))\vee(-0\wedge2\tilde{i}_u(\beta_3)-$ $\tilde{i}_u(\beta_3^2))]$
I_8	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i}))+0\wedge(\wedge_{i=1}^n\beta_{2i}(v))+0\wedge\tilde{i}_u(\beta_3)\wedge$ $(-\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i}))+0\vee(\vee_{i=1}^n\beta_{2i}(v))+0\vee\tilde{s}_u(\beta_3)\vee$ $(-\tilde{i}_u(\beta_3))]$
I_9	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i})+\wedge_{i=1}^n\beta_{2i}(v))\wedge\tilde{i}_u(\beta_3)]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i})+\vee_{i=1}^n\beta_{2i}(v))\vee\tilde{s}_u(\beta_3)]$
I_{10}	$(-1)\vee[0\wedge(\wedge_{i=1}^n\tilde{i}_u(\beta_{1i}))+0\wedge(\wedge_{i=1}^n\beta_{2i}(v))+0\wedge\tilde{i}_u(\beta_3)\wedge$ $(-\tilde{s}_u(\beta_3))]$	$1\wedge[0\vee(\vee_{i=1}^n\tilde{s}_u(\beta_{1i}))+0\vee(\vee_{i=1}^n\beta_{2i}(v))+0\vee\tilde{s}_u(\beta_3)\vee$ $(-\tilde{i}_u(\beta_3))]$

$(m = 1, 2, \ldots)$. If $\lim_{m\to\infty}\sup_{u\in U}|\beta_{11m}(u)| = \cdots = \lim_{m\to\infty}\sup_{u\in U}|\beta_{1nm}(u)| = 0$, $\lim_{m\to\infty}\sup_{v\in V}|\beta_{21m}(v)| = \cdots = \lim_{m\to\infty}\sup_{v\in V}|\beta_{2nm}(v)| = 0$, $\lim_{m\to\infty}\sup_{u\in U}|\beta_{3m}(u)| = 0$ hold, then we gain from Theorems 11–14 that $\lim_{m\to\infty} BKS(A_{1m}^*, \ldots, A_{nm}^*, B_{1m}^*, \ldots, B_{nm}^*, C_m^*)(v) = BKS(A_1, \ldots, A_n, B_1, \ldots, B_n, C)(v)$.

In sum, the BKS algorithm for the situation of multiple rules is stable from the perspective of simple perturbation.

Affective computing [37] is nowadays one of the most active research areas. Emotion deduction (which explores how to generate reasonable values of other emotions from some basic emotions) makes a critical difference in many aspects (e.g., when constructing massive emotional corpus, affect state transitions, and so forth), which is a vital task of affective computing. In what follows, we offer two examples of emotion deduction.

Example 1: We carry computing of emotion aspects via the BKS-I method where $\to$ is implemented as I_4. For the eight basic emotions, (including surprise, expect, anxiety, sorrow, angry, hate, joy, and love), we find that the former six emotions have a strong relationship with fear (as a new emotion). We establish the emotion deduction system from six basic emotions to fear. In detail, let $U = \{u_1, u_2, \ldots, u_6\}$ where $u_1 = 0, u_2 = 0.2, u_3 = 0.4, u_4 = 0.6, u_5 = 0.8, u_6 = 1.0$, and $V = \{v_1\}$ where $v_1 = 0.5$. Meanwhile, some rules from A_i to B_i and an input C are as follows:

$$A_1 = \frac{0.3}{u_1} + \frac{0.9}{u_2} + \frac{0.2}{u_3} + \frac{0.5}{u_4} + \frac{0.3}{u_5} + \frac{0.4}{u_6},\quad B_1 = \frac{0.2}{v_1}$$
$$A_2 = \frac{0.5}{u_1} + \frac{0.6}{u_2} + \frac{0.5}{u_3} + \frac{0.7}{u_4} + \frac{0.6}{u_5} + \frac{0.7}{u_6},\quad B_2 = \frac{0.4}{v_1}$$
$$A_3 = \frac{0.9}{u_1} + \frac{0.4}{u_2} + \frac{0.7}{u_3} + \frac{0.9}{u_4} + \frac{0.5}{u_5} + \frac{0.8}{u_6},\quad B_3 = \frac{0.6}{v_1}$$
$$A_4 = \frac{0.6}{u_1} + \frac{0.7}{u_2} + \frac{0.9}{u_3} + \frac{0.4}{u_4} + \frac{0.3}{u_5} + \frac{0.5}{u_6},\quad B_4 = \frac{0.9}{v_1}$$
$$C = \frac{0.8}{u_1} + \frac{0.7}{u_2} + \frac{0.6}{u_3} + \frac{0.9}{u_4} + \frac{0.5}{u_5} + \frac{0.8}{u_6}.$$

Here, the input A_i reflects the value for six basic emotions and the output B_i stands for the value for the emotion fear. For the input C, the BKS-I solution from (7) is $D(v_1) = \inf_{u\in U}\{C(u) \to \wedge_{i=1}^n(A_i(u) \to B_i(v_1))\} = 0.5 \wedge 0.3 \wedge 0.5 \wedge 0.4 \wedge 0.5 \wedge 0.4 = 0.3$.

For one situation, we take $\beta_{11}(u) = 0.05$, $\beta_{12}(u) = 0.06$, $\beta_{13}(u) = 0.07$, $\beta_{14}(u) = 0.08$, $\beta_{21}(v) = 0.06$, $\beta_{22}(v) = 0.07$, $\beta_{23}(v) = 0.08$, $\beta_{24}(v) = 0.09$, $\beta_3(u) = 0.05$ $(u \in U, v \in V)$. Let $A_i^*(u) = A_i(u) + \beta_{1i}(u)$, $B_i^*(v) = B_i(v) + \beta_{2i}(v)$, and $C^*(u) = C(u) + \beta_3(u)$ $(i = 1, 2, 3, 4)$.

From Theorem 11 for I_4, we obtain that the lower bound of $\Delta_n(v_1)$ is $[\wedge_{i=1}^n(-\tilde{s}_u(\beta_{1i}))] \wedge [\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})] \wedge [\wedge_{i=1}^n \beta_{2i}(v_1)] \wedge (-\tilde{s}_u(\beta_3)) \wedge \tilde{i}_u(\beta_3) = (-0.08) \wedge 0.05 \wedge 0.06 \wedge (-0.05) \wedge 0.05 = -0.08$. Analogously, the upper bound of $\Delta_n(v_1)$ is 0.09. Hence, we obtain $-0.08 \le \Delta_n(v_1) \le 0.09$ and $0.22 \le BKS(A_1^*, \ldots, A_4^*, B_1^*, \ldots, B_4^*, C^*)(v) \le 0.39$.

For another situation, we adopt $\beta_{1i}(u) = \beta_{2i}(v) = \beta_3(u) = 0.02$ $(u \in U, v \in V, i = 1, 2, 3, 4)$. Afterward, we can similarly gain $-0.02 \le \Delta_n(v_1) \le 0.02$ and $0.28 \le BKS(A_1^*, \ldots, A_4^*, B_1^*, \ldots, B_4^*, C^*)(v) \le 0.32$. Obviously, in this case, the value of $BKS(A_1^*, \ldots, A_4^*, B_1^*, \ldots, B_4^*, C^*)(v)$ is closer to $BKS(A_1, \ldots, A_4, B_1, \ldots, B_4, C)(v)$.

Example 2: We involve the BKS-T method in which $\to$ is realized as I_7 and $\otimes$ is taken as $\otimes_M$. We use the same $A_1, \ldots, A_4, B_1, \ldots, B_4$ as Example 1. New input is $C = (0.7/u_1)+(0.2/u_2)+(0.3/u_3)+(0.8/u_4)+(0.9/u_5)+(0.6/u_6)$. Then, the BKS-T solution from (6) is $D(v_1) = \inf_{u\in U}\{C(u) \to \vee_{i=1}^n(A_i(u) \otimes B_i(v_1))\} = 0.6 \wedge 0.96 \wedge 0.91 \wedge 0.6 \wedge 0.5 \wedge 0.64 = 0.5$.

For one case, we choose $\beta_{11}(u) = 0.08$, $\beta_{12}(u) = 0.07$, $\beta_{13}(u) = 0.06$, $\beta_{14}(u) = 0.07$, $\beta_{21}(v) = 0.07$, $\beta_{22}(v) = 0.06$, $\beta_{23}(v) = 0.06$, $\beta_{24}(v) = 0.05$, and $\beta_3(u) = 0.06$ $(u \in U, v \in V)$. Let $A_i^*(u) = A_i(u) + \beta_{1i}(u)$, $B_i^*(v) = B_i(v) + \beta_{2i}(v)$, and $C^*(u) = C(u) + \beta_3(u)$ $(i = 1, 2, 3, 4)$.

In the light of Theorem 12 for I_7, we have that the lower bound of $\Delta_n(v_1)$ is $(-1) \vee [(\wedge_{i=1}^n \tilde{i}_u(\beta_{1i})) \wedge (\wedge_{i=1}^n \beta_{2i}(v_1)) \wedge$

$(-0 \vee 2\tilde{s}_u(\beta_3) - \tilde{s}_u(\beta_3^2))] = (-1) \vee [0.06 \wedge 0.05 \wedge (-0 \vee 0.12 - 0.06^2)] = -0.1236$. Analogously the upper bound of $\Delta_n(v_1)$ is 0.08. Consequently, we find $-0.1236 \leq \Delta_n(v_1) \leq 0.08$ and $0.3764 \leq BKS(A_1^*, \ldots, A_4^*, B_1^*, \ldots, B_4^*, C^*)(v) \leq 0.58$.

For another case, we let $\beta_{1i}(u) = \beta_{2i}(v) = \beta_3(u) = 0.02$ $(u \in U, v \in V, i = 1, 2, 3, 4)$. In a similar way we obtain $-0.0404 \leq \Delta_n(v_1) \leq 0.02$ and $0.4596 \leq BKS(A_1^*, \ldots, A_4^*, B_1^*, \ldots, B_4^*, C^*)(v) \leq 0.52$. In this case, $BKS(A_1^*, \ldots, A_4^*, B_1^*, \ldots, B_4^*, C^*)(v)$ evidently achieves a closer result to $BKS(A_1, \ldots, A_4, B_1, \ldots, B_4, C)(v)$.

VI. Discussion

1) Definition 9 (i.e., Pappis's proximity measure [17]), Definition 10 (i.e., Hong and Hwang' ε-similarity [18]), and Definition 12 (viz., Cai's δ-equality [34]) can be incorporated into the framework of interval perturbation of fuzzy sets. Moreover, Definition 11 (i.e., Ying's maximum ε perturbation of a fuzzy set in [19]) can be deemed as a special situation of simple perturbation. Then, the interval perturbation and simple perturbation are more general ways to express the robust issue. So this work is more meaningful vis-à-vis the previous studies.
2) In this study, we obtain the upper and lower bounds of BKS output deviation derived from the perturbation of the input fuzzy set, which provide more detailed characterization for the output deviation along with the input perturbation.
3) The previous works have exhibited that the BKS method has good performance, including the interpolativity, continuity, robustness, and that the BKS-based FRI is as effective and efficient as the CRI method (see [7], [12], and [13]). As their development, this work goes a step further to validate the stability of BKS from the perspective of oscillation-bound estimation of perturbations, which is also as good as the CRI method.
4) In [29], the CRI-based FRI with varying limits of input is researched. Our work in this study for the BKS-based FRI has three disparate and original points with [29] as follows.
   a) The work in [29] only focuses on modeling form $\check{R}$ in CRI, where the CRI solution is $D(v) = \sup_{u \in U}\{C(u) \otimes R(u, v)\} = \sup_{u \in U}\{C(u) \otimes (A(u) \to B(v))\}$ $(v \in V)$. Thereinto, $\check{R}_1(u, v) = A(u) \to B(v)$ is adopted. However, in this study, we discover the situations of both $\check{R}$ (using $\to$) and $\hat{R}$ (employing $\otimes$) in BKS. Hence, our modeling strategies are more comprehensive.
   b) The work in [29] only aims at one rule in CRI. That is, $\check{R}_1(u, v) = A(u) \to B(v)$ is considered. By our findings in this study, the performance of BKS is as good as CRI for the situation of $\check{R}_1$. But more importantly, we make a thorough inquiry into the cases of both one rule and multiple rules for BKS. To be specific, four kinds of situations are revealed, which incorporate $\check{R}_1(u, v) = A(u) \to B(v)$, $\hat{R}_2(u, v) = A(u) \otimes B(v)$, $\check{R}_3(u, v) = \vee_{i=1}^n \{A_i(u) \otimes B_i(v)\}$, and $\hat{R}_4(u, v) = \wedge_{i=1}^n \{A_i(u) \to B_i(v)\}$ $(u \in U, v \in V)$. Notice that the results of BKS for the last three are also good.
   c) The chain of fuzzy reasoning has the problem of error propagation since it consists of multiple nested inferences. Consequently, it has a stricter demand for stability and possesses more important value for the stability research than the ordinary fuzzy reasoning. The work in [29] does not consider the problem of the fuzzy reasoning chain. However, this study explores such challenging problem and has a confirmed the stability of BKS chain reasoning.

VII. Conclusion

It should be stressed that the previous works mainly concentrated on how the output values were altered due to perturbation parameters of input values. However, studies on estimating oscillation bounds of output values with respect to varying limits of input values were not available. This study directed at this problem and offered the upper and lower bounds of BKS output, in which the two bounds are characterized in the form of fuzzy sets. Then, the stable (or asymptotically stable) properties are verified for the BKS algorithms. We systematically investigated the oscillation-bound estimation of perturbations for the fuzzy reasoning method of BKS. The key contributions are as follows.

Above all, we focus on two kinds of BKS algorithms, (including the BKS-T and BKS-I algorithms corresponding to two different modeling methods for the rule base) with one rule as well as multiple rules, and research the situation of interval perturbation. Corresponding upper and lower bounds of BKS output variation are offered, where the R-implication, (S, N)-implication, QL-implication, and t-norm implication are adopted. The stability of the BKS method for interval perturbation has been validated.

Furthermore, focusing on the chain of fuzzy reasoning with BKS, the BKS output scope originating from input interval perturbation are given, where the R-implication, (S, N)-implication, QL-implication, and t-norm implication are utilized. The stability of BKS chain reasoning has been confirmed.

In the end, we investigated the situation of simple perturbation for the BKS algorithm with one rule and multiple rules and obtained the upper and lower bounds of BKS output deviation derived from the simple perturbation of the input fuzzy set, in which ten fuzzy implications, (including some representative R-implications, (S, N)-implications, QL-implications, and t-norm implications) together with three t-norms are employed. The stable and asymptotic stable properties of these BKS strategies are verified.

It is noted that the interval perturbation and simple perturbation are more general expression ways for the robust issue, hence the work for the interval perturbation and simple perturbation is more meaningful than the previous works. In addition, the obtained oscillation bound gives more detailed characterization for the output deviation along with the input perturbation. This work goes a step further to validate the good properties of the BKS method of fuzzy reasoning, which is as good as the CRI method.

Here, we emphasize the novelty of this study.

1) The oscillation-bound estimation of perturbations is a novel problem to be explored for BKS (noting that the foregoing works in relation to stability and robustness of BKS mainly focused on how the output values were altered due to perturbation parameters of input values).
2) Directing at fuzzy chain reasoning with BKS, its output scope resulting from input interval perturbation is offered. Note that such a point for the chain of FRI was lacking in the previous research.
3) We explore the error estimation of perturbations for both $\check{R}$ and $\hat{R}$ in the BKS scheme. It is noticed that such a topic for the situation of $\hat{R}$ was not investigated in the foregoing studies on FRI.

In future studies, it is worth developing the BKS method from the perspective of granular computing (see [38] and [39]) and studying on how to construct and design rational fuzzy controllers by choosing some ideal BKS strategies.

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