Granular Symmetric Implicational Method

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***Abstract*—As a means of describing realistic problems, fuzzy sets can be included into the category of information granules from a broader perspective. Then the interval-valued fuzzy set itself is an expression of information granule which has more elaborate and stronger characterization abilities than generic fuzzy sets. In this study, facing up with modeling situations involving the use of interval-valued fuzzy sets, we come up with the granular symmetric implicational (GSI) method of fuzzy inference in view of the symmetric implicational idea and granular computing, which includes the basic GSI method and the $\zeta(w, z)$-GSI method. First, complete residuated lattices are employed as the structures of truth-values for interval-valued fuzzy sets. Second, unified expressions of optimal solutions to two GSI methods are gained for R-implications and (S, N)-implications. Lastly, it is shown through examples that the GSI method is superior over corresponding interval-valued fully implicational method. The originality of this work is three-fold. To begin with, the interval-valued fuzzy operators are introduced to the symmetric implicational mechanism, and novel symmetric implicational principles are presented which ameliorate the previous ones. Moreover, we offer a new construction method for interval-valued implications and corresponding adjoint couples, and on the strength of it we validate the reversibility and continuous properties of the GSI method. Finally, the hierarchical granular inference strategy is established for the GSI method in allusion to the circumstance of multiple rules.**

***Index Terms*—Granular computing, fuzzy inference, fuzzy implication, compositional rule of inference, fully implicational method.**

I. Introduction

FUZZY inference is an advanced intelligent computing framework based on the concepts of fuzzy set theory, fuzzy if-then rules and approximate inference. Currently fuzzy inference comes with an extensive theory and applications in the fields of fuzzy control, machine learning, affective computing, and humanoid robot [1]–[3]. Its essential connotation is

$$\text{If } P \implies Q, \text{ then } P^\diamond \implies Q^\diamond, \tag{1}$$

which is in relation to the FMP (fuzzy modus ponens) problem:

$$\text{FMP: for } P \Longrightarrow Q \text{ and } P^\diamond, \text{deduce } Q^\diamond, \tag{2}$$

in which $P, P^\diamond \in \Phi(W)$, $Q, Q^\diamond \in \Phi(Z)$ while $\Phi(W), \Phi(Z)$ reflect the set of entire fuzzy subsets of universes W, Z.

A. The CRI Method and the Symmetric Implicational Method

Gaining insight into the internal mechanism of FMP, an implication $\rightarrow$ was adopted to characterize $\Longrightarrow$, and then the compositional rule of inference (CRI) method was found by Zadeh [4]. The solution of CRI comes as $Q^\diamond(z) = \sup_{w\in W}\{P^\diamond(w) \wedge (P(w) \rightarrow Q(z))\},\ \ z \in Z.$

Then single implication was extended to triple ones. To connect closely fuzzy reasoning and fuzzy formal deduction theory, Wang [5] proposed the fully implicational method to set a logic foundation for developing the theory of fuzzy reasoning. As for its internal mechanism, the ideal solution of FMP is acquired from the major premise $P \rightarrow Q$ and the minor premise $P^\diamond$ together, and $P^\diamond \Longrightarrow Q^\diamond$ should be considered (with $P^\diamond \rightarrow Q^\diamond$) and should be adequately supported by $P \rightarrow Q$, where supporting is also characterized by $\rightarrow$. Then the structure of three implications is formed, and its ideal solution is the smallest $Q^\diamond \in \Phi(Z)$ making

$$(P(w) \rightarrow Q(z)) \rightarrow (P^\diamond(w) \rightarrow Q^\diamond(z)) \tag{3}$$

attain its maximum for any $w \in W, z \in Z$. Song *et al.* [6] researched the fully implicational method for Zadeh implication I_Z, and the corresponding reversibility properties were analyzed. Wang and Fu [7] established unified forms of the fully implicational method of which diverse implications can be used, and pointed out that the CRI method could be brought into line with these unified forms. Liu and Wang [8] validated the continuity of the fully implicational method, including the cases for the R-implications and the Zadeh implication. Pei [9] built a sound logical foundation for the fully implicational method with verifying its consistency, which was on the strength of a monoidal t-norm based logical system. Li and Liu [10] analyzed the entire fully implicational method for double fuzzy control systems and manifold learning of dimensionality reduction, and discussed their reversibility. Zheng and Liu [11] investigated the fully implicational method on intuitionistic fuzzy sets, in which the corresponding multiple-rules models were established. Luo and Liu [12] verified the robustness of interval-valued fully implicational method on account of normalized

Manuscript received April 5, 2021; revised June 28, 2021; accepted July 9, 2021. Date of publication August 10, 2021; date of current version May 26, 2022. This work was supported in part by the National Key Research and Development Program of China under Grant 2020YFC1523100, and in part by the National Natural Science Foundation of China under Grants 61673156, 61877016, 61976078, and 61672202. *(Corresponding author: Yiming Tang.)*

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Digital Object Identifier 10.1109/TETCI.2021.3100597

Minkowski distance. Luo and Wang [13] researched the interval-valued fully implicational method based on the left-continuous t-representable t-norm, and its reversibility property and robustness were verified. Following systematic discovery ([6]–[13]), it was validated that the fully implicational method exhibited many favorable properties including logic basis, reversibility, robustness among others.

In the fully implicational method, three implications are same. In fact, implication connective is the fundament for a logic system to carry through inference. The implication in (3) is to form an intimate contact between the fuzzy inference and the logic system, thus gives fine logic basis for fuzzy inference. Analyzing (1) and (3), the first and third implications in (3) can be viewed as the implication connective in a logic system; and the second implication in (3) represents the "if-then" relationship of the model (1). So, in [14], we came up with a novel fuzzy inference method called the symmetric implicational method, which generalized (3) to

$$(P(w) \to_1 Q(z)) \to_2 (P^{\diamond}(w) \to_1 Q^{\diamond}(z)). \tag{4}$$

Here $\to_1$ and $\to_2$ can adopt different implications, which means that they can be the same or disparate. As for the implications used in (4), we prefer R-implications or (S,N)-implications. In [14], we developed the basic principles of the symmetric implicational method, and offered its ideal solutions, while its reversibility was proved. Then, we presented the $\zeta(w,z)$-symmetric implicational method in [15], where (4) was transformed into

$$(P(w) \to_1 Q(z)) \to_2 (P^{\diamond}(w) \to_1 Q^{\diamond}(z)) \geq \zeta(w,z), \tag{5}$$

where $\zeta(w,z)$ was a mapping from $W \times Z$ to [0,1], which reflected the idea of two-dimensional sustaining degree. In [15], its ideal solutions were gained, and it was found that the $\zeta(w,z)$-symmetric implicational method made all symmetric implicational methods and fully implicational methods form a unified whole. Dai [16] provided a predicate formal representation and logic proof of the solutions to the symmetric implicational method based upon the $L\Pi$ logic and formalized its consistency on the strength of a formal logic system $L\Pi\forall$. This brought the symmetric implicational method within a logical framework and provided a sound logic foundation for the symmetric implicational method.

B. Motivation

To sum up, new tendency of fuzzy inference mentioned above lies in the following aspects. First, the exploration of fuzzy inference for specific implications ([5], [6]) has been developed into the analyses for certain kinds of implications ([7]–[16]). Second, the researches on properties always occupy a high position ([8]–[16]) because they are important criterions to judge whether a fuzzy inference method is effective or not. Third, using the same implications ([5]–[13]) has been upgraded to employ different implications ([14]–[16]). Lastly, the ordinary fuzzy environment ([5]–[10], [14]–[16]) has been evolved into more complex one ([11]–[13]).

As for the symmetric implicational method, our previous works only aimed at ordinary fuzzy environment. Under the new tendency, it is worth exploring it in more complex environment. We show the details below.

Zadeh proposed fuzzy sets to deal with the aspect of uncertainty found in the definition of a vagueness concept or meaning, which has made a groundbreaking and outstanding contribution to the field. The fuzzy set was characterized by the mapping $P : W \to [0,1]$ $(w \in W)$. From the viewpoint of granular computing [17], such specific number can be extended to a broader concept of information granule [18], [19] (e.g., an interval). This consideration has resulted in some extensions of fuzzy sets. For instance, the interval-valued fuzzy set [20] was an important extension (see Lemma 1.1 (i) in what follows), which was characterized by the mapping $\mathcal{P} : W \to [0_L, 1_L]$, where $0_L = [0,0], 1_L = [1,1]$.

Lemma I.1: (i) The interval-valued fuzzy set can be regarded as an extension of the fuzzy set. (ii) $\mathcal{P}(w) \equiv [b^-, b^+]$ $(w \in W)$ is an interval-valued fuzzy set on W. (iii) For any interval membership degree $[b^-, b^+]$ to depict an element with respect to (w.r.t. for short) a linguistic term, if its interval length is larger, then the uncertainty expressed by $[b^-, b^+]$ is higher; and vice versa. (iv) For any interval membership degree $[b^-, b^+]$ to depict an element w.r.t. a linguistic term, b^- reflects the smallest numerical value to depict this element, while b^+ embodies the largest numerical value.

Proof: (i) From the definition of the interval-valued fuzzy set (i.e., Definition 2.9 in what follows), note that an interval-valued fuzzy set can be represented by $\mathcal{P}(w) = [\mu_{\mathcal{P}}(w), \nu_{\mathcal{P}}(w)] \in L$ $(w \in W)$. If $\mu_{\mathcal{P}}(w) \equiv \nu_{\mathcal{P}}(w)$ $(w \in W)$, then the interval-valued fuzzy set $\mathcal{P}$ can be regarded as a fuzzy set. From this viewpoint, the interval-valued fuzzy set can be regarded as an extension of the fuzzy set. (ii) For an interval-valued fuzzy set $\mathcal{P}(w) = [\mu_{\mathcal{P}}(w), \nu_{\mathcal{P}}(w)] \in L$, if $\mu_{\mathcal{P}}(w) \equiv b^-$ and $\nu_{\mathcal{P}}(w) \equiv b^+$ $(w \in W)$, then $\mathcal{P}(w) \equiv [b^-, b^+]$ $(w \in W)$, which is obviously a special interval-valued fuzzy set on W. (iii) For any interval membership degree $[b^-, b^+]$ to characterize an element w.r.t. a linguistic term, we denote its interval length $len = b^+ - b^-$. If the interval length len is larger, then this reflects the greater uncertainty expressed by $[b^-, b^+]$ about the characterization of the element, since it is even harder to determine what the exact number is. (iv) For any interval membership degree $[b^-, b^+]$ to characterize an element w.r.t. a linguistic term, the left interval endpoint b^- is obviously the smallest numerical value to depict the element w.r.t. the linguistic term. On the contrary, the right interval endpoint b^+ is the largest numerical value. ■

On the one hand, intervals can be reconstructed as interval-valued fuzzy sets. In fact, aiming at an interval $[b^-, b^+]$, we can construct $\mathcal{P}(w) \equiv [b^-, b^+]$ $(w \in W)$ from the viewpoint of Lemma 1.1(i), which is an interval-valued fuzzy set on W. On the other hand, interval membership degrees can be used to represent the uncertainty to precisely determine the proper membership degree of an element w.r.t. a linguistic term, as considered in interval-valued fuzzy sets. In one case, the interval length is used to provide an estimation of the uncertainty during membership assignment (see Lemma 1.1(iii)). Interval values can also be viewed as summarizing the opinions of several experts about the exact membership degree for an element w.r.t. a linguistic term. In another case, the left and right interval endpoints are, respectively, the smallest and largest numerical values provided by experts (see Lemma 1.1(iv)). In both cases, the richness of interval structures provides tools to deal with such notions of uncertainty.

As a result, under the environment of granular computing, we hope to establish a new framework to fuzzy inference, i.e., the granular fuzzy inference. As an exploration of this topic, in this study, we use the symmetric implicational method as its basis, and come up with a novel fuzzy inference method called the granular symmetric implicational (GSI) method. It aims at (4) and (5) where $\rightarrow_1, \rightarrow_2$ can employ different interval-valued implications and $P, Q, P^\diamond, Q^\diamond$ are interval-valued fuzzy sets, which are in turn said to be the basic GSI method and the $\zeta(w,z)$-GSI method. The purpose of this work is to research the GSI method.

Section II covers the preliminaries. Section III presents the basic $\zeta(w,z)$-GSI method revolving around its principle, solving strategy and its hierarchical idea. Section IV investigates its developed algorithm, i.e., the $\zeta(w,z)$-GSI method. Section V discusses their reversibility and the continuous properties. Section VI offers some examples for the $\zeta(w,z)$-GSI method. Section VII summarizes the entire work.

II. Preliminaries

Definition II.1: ([21]) (i) A function $T : [0,1]^2 \rightarrow [0,1]$ is referred to as a triangular norm (abbreviated by t-norm), if it is commutative, associative, increasing, and owns the neutral point 1 (i.e., $T(w,1) = w$ holds for any $w \in [0,1]$).

(ii) A function $S : [0,1]^2 \rightarrow [0,1]$ is called a triangular conorm (abbreviated by t-conorm), if it is commutative, associative, increasing, and owns the neutral point 0 (i.e., $S(w,0) = w$ holds for any $w \in [0,1]$).

Definition II.2: ([21]) A decreasing funtion $N : [0,1] \rightarrow [0,1]$ goes by the name of a fuzzy negation whenever $N(1) = 0$, $N(0) = 1$ works. It is referred to as a strong negation when $N(N(w)) = w$ comes into existence ($w \in [0,1]$). And N is said to be strict when it is continuous and strictly decreasing.

$N_s(w) = 1 - w\,(w \in [0,1])$ goes by the name of the standard negation on [0,1]. Apparently N_s is a strong negation and a strict negation. $N_K(w) = 1 - w^2$ is strict. $N_{D1}(w) = \begin{cases} 1, & w = 0 \\ 0, & w > 0 \end{cases}$ and $N_{D2}(w) = \begin{cases} 1, & w < 1 \\ 0, & w = 1 \end{cases}$ are the least and greatest ones, and are non-strong.

Definition II.3: ([21]) *The antithesis of a t-norm T (t-conorm S) on [0,1] w.r.t. a strong negation N is the function T_N (S_N) which is represented as $T_N(w,z) = N(T(N(w), N(z)))$ ($S_N(w,z) = N(S(N(w), N(z)))$) where $w, z \in [0,1]$.*

Definition II.4: ([22]) An implication on [0,1] is a function $I : [0,1]^2 \rightarrow [0,1]$ making three properties be effective:

(PR1) $I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$,

(PR2) $I(w,r) \geq I(z,r)$ if $w \leq z$,

(PR3) $I(w,z) \geq I(w,r)$ if $z \geq r$.

$I(w,z)$ is also recorded as $w \rightarrow z$ ($w, z, r \in [0,1]$).

From Definition 2.4, the following relationship works for I.

(PR4) $I(0,w) = I(w,1) = 1$ ($w \in [0,1]$)

Definition II.5: ([23]) Let T, I be two $[0,1]^2 \rightarrow [0,1]$ functions, (T,I) is referred to as an adjoint couple, whenever $T(w,z) \leq r \iff z \leq I(w,r)$ is true ($w, z, r \in [0,1]$).

Take notice of that the function T adjoint to I is one and only, and the reverse is also true.

Lemma II.1: ([14]) Let I be an implication on [0,1], if (PR5) $I(w,z)$ is right-continuous w.r.t. z, is effective, then $T : [0,1]^2 \rightarrow [0,1]$ indicated as

$$T(w,z) = \inf\{y \in [0,1] \mid z \leq I(w,y)\}, \quad w, z \in [0,1] \tag{6}$$

is adjoint to I.

Definition II.6: ([24]) $I : [0,1]^2 \rightarrow [0,1]$ is referred to as an R-implication, whenever a left-continuous t-norm T exists and makes $I(w,z) = \sup\{y \in [0,1] \mid T(w,y) \leq z\}$, $w, z \in [0,1]$.

Definition II.7: ([22], [25]) A function $I : [0,1]^2 \rightarrow [0,1]$ is known as an (S, N)-implication, whenever there are a t-conorm S and a fuzzy negation N making $I(w,z) = S(N(w), z)$, $w, z \in [0,1]$. In addition, if N is a strong negation, then I goes by the name of an S-implication.

Definition II.8: ([26]) If I is an implication, then the function N_I: $[0,1] \rightarrow [0,1]$ indicated by $N_I(w) = I(w,0)$ ($w \in [0,1]$) is referred to as the natural negation of I.

Lemma II.2: ([7], [25]) Assume that I is an R-implication from a left-continuous t-norm T, then (T,I) is an adjoint couple, and I meets (PR1),(PR2),(PR3),(PR4),(PR5) and

(PR6) $I(0,w) = 1$

(PR7) $I(1,w) = w$,

(PR8) $I(w,z)$ is left-continuous w.r.t. w,

(PR9) $w \leq z \iff I(w,z) = 1$,

(PR10) $I(w, I(z,r)) = I(z, I(w,r))$,

(PR11) $I(T(w,z), r) = I(w, I(z,r))$,

(PR12) $w \leq I(z,r) \iff z \leq I(w,r)$,

(PR13) $I(\sup_{y \in Y} y, w) = \inf_{y \in Y} I(y,w)$,

(PR14) $I(w, \inf_{y \in Y} y) = \inf_{y \in Y} I(w,y)$.

Among them $w, z, r, y \in [0,1]$ and $Y \subset [0,1]$, $Y \neq \varnothing$.

Proposition II.1: ([25], [26]) If I is an (S, N)-implication constructed by a fuzzy negation N and a t-conorm S, then I meets (PR6), (PR7), (PR10) and

(PR15) $N = N_I$.

In addition, any (S,N)-implication I meets ($w, z \in [0,1]$)

(PR16) $I(w,z) = I(N(z), N(w))$,

iff I is an S-implication.

Definition II.9: ([27], [28]) An IFS on W is a function $\mathcal{P} : W \rightarrow L$, $\mathcal{P}(w) = [\mu_{\mathcal{P}}(w), \nu_{\mathcal{P}}(w)] \in L$ ($w \in W$), where $L = \{[e^-, e^+] \mid e^-, e^+ \in [0,1],\ e^- \leq e^+\}$. The order $\leq_L$ on L is characterized by $[e^-, e^+] \leq_L [f^-, f^+] \iff e^- \leq f^-,\ e^+ \leq f^+$. In addition, $[e^-, e^+] \leq_L [f^-, f^+]$ is also represented by $[f^-, f^+] \geq_L [e^-, e^+]$. And the set of all IFSs on W is also indicated by $\Phi(W)$. In allusion to any nonempty subset C of L, infimum ($\wedge$) and supremum ($\vee$) are indicated as $\wedge\{c \mid c \in C\} = [\wedge\{c^- \mid c \in C\},\ \wedge\{c^+ \mid c \in C\}]$ and $\vee\{c \mid c \in C\} = [\vee\{c^- \mid c \in C\},\ \vee\{c^+ \mid c \in C\}]$.

$(L, \wedge, \vee, 0_L, 1_L)$ is a complete lattice [27], where $0_L = [0,0]$ and $1_L = [1,1]$ are the smallest and greatest value, in turn. An IFS can be seen as an L-fuzzy set from the perspective of Goguen [28] w.r.t. L. We adopt complete residuated lattices as the structures of truth-values for interval-valued fuzzy sets.

Definition II.10: ([29]) A residuated lattice is an algebra $\Omega = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *meeting*

(i) $(L, \wedge, \vee, 0, 1)$ *is a lattice with the least value 0 and the greatest value 1,*

(ii) $(L, \otimes, 1)$ *is a commutative monoid with the unit 1, i.e.,* $\otimes$ *is commutative, associative and* $w \otimes 1 = w$ $(w \in L)$*,*

(iii) $\otimes$ *and* $\rightarrow$ *construct an adjoint couple, i.e.,* $w \otimes z \leq r \iff z \leq w \rightarrow r$ *holds for all* $w, z, r \in L$*.*

In addition, if $(L, \wedge, \vee, 0, 1)$ *is a complete lattice, then* Ω *is referred to as a complete residuated lattice.*

In Definition 2.10, $\otimes$ (called multiplication) and $\rightarrow$ (called residuum), respectively aim to model the conjunction and implication of the corresponding logical calculus, while supremum ($\vee$) and infimum ($\wedge$) are adopted for signifying the existential and general quantifier.

The lattice $(L, \wedge, \vee, 0_L, 1_L)$ from Definition 2.9, is already a complete lattice. For this reason, we put emphasis on the properties of residuated lattices, where the key one lies in the adjunction property.

Definition II.11: ([30]) (i) The interval-valued fuzzy t-norm $\mathcal{T}$ *on L is an associative, increasing, commutative function, and owns the neutral element* 1_L *(i.e.,* $\mathcal{T}(w, 1_L) = w$ *holds for any* $w \in L$*).*

(ii) The interval-valued fuzzy t-conorm $\mathcal{S}$ *on L is an associative, increasing, commutative function, and possesses the neutral element* 0_L *(i.e.,* $\mathcal{S}(w, 0_L) = w$ *holds for any* $w \in L$*).*

Definition II.12: Assume that W *is a non-empty set, then the partial order relation* $\leq_\Phi$ *on* $\Phi(W)$ *is indicated as* $P \leq_\Phi Q$ *iff* $P(w) \leq_L Q(w)$ $(w \in W)$*, in which* $P, Q \in \Phi(Z)$*.*

It is easy to see that $< \Phi(Z), \leq_\Phi >$ is a complete lattice.

Definition II.13: ([20]) The interval-valued implication $\mathcal{I}$ *is a function* $L^2 \rightarrow L$ *meeting:*

(P1) $\mathcal{I}(0_L, 0_L) = \mathcal{I}(1_L, 1_L) = 1_L$*, and* $\mathcal{I}(1_L, 0_L) = 0_L$*.*

(P2) $\mathcal{I}(w, z) \geq_L \mathcal{I}(w, r)$ *if* $z \geq_L r$ $(w, z, r \in L)$*,*

(P3) $\mathcal{I}(w, r) \geq_L \mathcal{I}(z, r)$ *if* $w \leq_L z$ $(w, z, r \in L)$*.*

In addition, $\mathcal{I}(w, z)$ *is also signified by* $w \rightarrow z$ $(w, z \in L)$*.*

From Definition 2.13,

(P4) $\mathcal{I}(0_L, w) = \mathcal{I}(w, 1_L) = 1_L$ $(w \in L)$.

meets for any interval-valued implication $\mathcal{I}$.

The following definition offers a construction strategy for a t-norm on L.

Definition II.14: ([31]) For a t-norm T on [0,1], the $L^2 \rightarrow L$ *function represented by* $(w, z \in L)$*:*

$$\mathcal{T}_T(w, z) = [T(w^-, z^-), \max\{T(w^-, z^+), T(w^+, z^-)\}]$$

is a t-norm on L, which is known as the pessimistic t-norm with representative T.

The next proposition affords a new kind of interval-valued implication, which leads to a method to construct an implication on L from an implication on [0,1].

Proposition II.2: Assume that I is an implication on [0,1] meeting (PR5), and that T signified by (6) is its adjoint function, and that $p \in [0, 1]$*. Then the function* $\mathcal{I}_{I,p} : L^2 \rightarrow L$ *represented by* $(w, z \in L)$

$$\mathcal{I}_{I,p}(w, z) = [\min\{I(w^-, z^-),\ I(w^+, z^+)\},\ \min\{I(T(p, w^+), z^+),\ I(w^-, z^+)\}] \quad (7)$$

is referred to as an interval-valued implication on L.

Proof: (i) The implication I meets (PR5), so one has from Lemma 2.1 that T indicated by (6) is adjoint to I. Therefore we gain $T(w, 0) = \inf\{y \in [0, 1] \mid 0 \leq I(w, y)\} = 0$ for any $w \in [0, 1]$. It implies that $\mathcal{I}_{I,p}(0_L, 0_L) = [\min\{1, 1\}, \min\{I(0, 0),\ 1\}] = [1, 1] = 1_L$. Analogously one has $\mathcal{I}_{I,p}(1_L, 0_L) = 0_L$ and $\mathcal{I}_{I,p}(0_L, 1_L) = \mathcal{I}_{I,p}(1_L, 1_L) = 1_L$.

(ii) Assume that $w, z, r \in L$, and $z \geq_L r$. Because I meets (PR3), it is straightforward to gain $I(w^-, z^-) \geq I(w^-, r^-)$, $I(w^+, z^+) \geq I(w^+, r^+)$, $I(T(p, w^+), z^+) \geq I(T(p, w^+), r^+)$, $I(w^-, z^+) \geq I(w^-, r^+)$. In consequence, $\mathcal{I}_{I,p}(w, z) \geq_L \mathcal{I}_{I,p}(w, r)$.

(iii) From (6), it is straightforward to acquire T is increasing w.r.t its second variable. Suppose that $w, z, r \in L$, and $w \leq_L z$. Noting I satisfies (PR2), one has $I(w^-, r^-) \geq I(z^-, r^-)$, $I(w^+, r^+) \geq I(z^+, r^+)$, $T(p, w^+) \leq T(p, z^+)$, $I(T(p, w^+), r^+) \geq I(T(p, z^+), r^+)$, $I(w^-, r^+) \geq I(z^-, r^+)$. For this reason, we gain $\mathcal{I}_{I,p}(w, r) \geq_L \mathcal{I}_{I,p}(z, r)$.

As a consequence, $\mathcal{I}_{I,p}$ indicated by (7) is an interval-valued implication on L. ■

Gasse *et al.* [32] also provided similar interval-valued implication as (7), however it was derived from the left-continuous t-norm. Next we easily deduce Corollary 2.1.

Corollary 2.1. *If I is an implication on [0,1] meeting (PR5), and T denoted by (6) is its adjoint function, and* $p = 0$*. Then the implication* $\mathcal{I}_{I,p}(w, z)$ *signified by (7) is* $(w, z \in L)$*:*

$$[\min\{I(w^-, z^-), I(w^+, z^+)\},\ I(w^-, z^+)]. \quad (8)$$

In truth, $\mathcal{I}_{I,p}$ represented by (8) can be deemed as the optimistic implication on L as one of the pseudo-i-representable implications [33].

Definition II.15: Let $\mathcal{T}$ *and* $\mathcal{I}$ *be two* $L^2 \rightarrow L$ *functions,* $(\mathcal{T}, \mathcal{I})$ *is referred to as an interval-valued adjoint couple, whenever the equivalent relationship is true* $(w, z, r \in L)$*:*

$$\mathcal{T}(w, z) \leq_L r \iff z \leq_L \mathcal{I}(w, r). \quad (9)$$

The next Proposition 2.3 shows a method to build an adjoint couple from the interval-valued fuzzy viewpoint.

Proposition II.3: Assume that I is an implication on [0,1] meeting (PR5), and that T signified by (6) is adjoint to I, and that $p \in [0, 1]$*. The function* $\mathcal{T}_{T,p} : L^2 \rightarrow L$ *is expressed as* $\mathcal{T}_{T,p}(w, z) = \inf\{\gamma \in L \mid z \leq_L \mathcal{I}_{I,p}(w, \gamma)\}$ $(w, z \in L)$*. Then we gain*

(i) $\mathcal{T}_{T,p}$ *can be indicated as below* $(w, z \in L)$*:*

$$\mathcal{T}_{T,p}(w, z) = [T(w^-, z^-), \max\{T(w^-, z^+), T(w^+, z^-),\ T(T(p, w^+), z^+)\}], \quad (10)$$

(ii) $\mathcal{T}_{T,p}$ *is adjoint to* $\mathcal{I}_{I,p}$ *which is represented by (7);*

(iii) $\mathcal{I}_{I,p}$*, which is adjoint to* $\mathcal{T}_{T,p}$*, is expressed as* $\mathcal{I}_{I,p}(w, z) = \sup\{\gamma \in L \mid \mathcal{T}_{T,p}(w, \gamma) \leq_L z\}$.

Proof: (i) Since the implication I meets (PR5), one gets from Lemma 2.1 that T denoted as (6) is adjoint to I. So (6) works for (T, I). Take into consideration that $\mathcal{I}_{I,p}(w, \gamma) = [\min\{I(w^-, \gamma^-),\ I(w^+, \gamma^+)\},\ \min\{I(T(p, w^+), \gamma^+),\ I(w^-, \gamma^+)\}]$.

Then we have ($w, z, \gamma \in L$)

$$z^+ \leq (\mathcal{I}_{I,p}(w,\gamma))^+$$
$$\Longleftrightarrow z^+ \leq \min\{I(T(p,w^+),\gamma^+), I(w^-,\gamma^+)\}$$
$$\Longleftrightarrow z^+ \leq I(T(p,w^+),\gamma^+)\ ,\ z^+ \leq I(w^-,\gamma^+)$$
$$\Longleftrightarrow T(T(p,w^+),z^+) \leq \gamma^+\ ,\ T(w^-,z^+) \leq \gamma^+.$$

In a similar manner, the following relationship is constructed.

$$z^- \leq (\mathcal{I}_{I,p}(w,\gamma))^- \Longleftrightarrow T(w^-,z^-) \leq \gamma^-,\ T(w^+,z^-) \leq \gamma^+.$$

For this reason, $\mathcal{T}_{T,p}$ can be changed into $\mathcal{T}_{T,p}(w,z) = \inf\{\gamma \in L \mid z \leq_L \mathcal{I}_{I,p}(w,\gamma)\} = [T(w^-,z^-),\ \max\{T(w^-,z^+), T(w^+,z^-), T(T(p,w^+),z^+)\}]$. That is, (10) works.

(ii) In consideration of the definition of $\leq_L$, Proposition 2.2 together with (7) and (10), noting that (6) is effective for (T, I), one has ($w, z, r \in L$)

$$\mathcal{T}_{T,p}(w,z) \leq_L r$$
$$\Longleftrightarrow T(w^-,z^-) \leq r^-\ ,$$
$$\max\{T(w^-,z^+), T(w^+,z^-), T(T(p,w^+),z^+)\} \leq r^+$$
$$\Longleftrightarrow z^- \leq I(w^-,r^-),\ T(w^-,z^+) \leq r^+,\ T(w^+,z^-) \leq r^+\ ,$$
$$T(T(p,w^+),z^+) \leq r^+$$
$$\Longleftrightarrow z^- \leq I(w^-,r^-), z^+ \leq I(w^-,r^+),\ z^- \leq I(w^+,r^+)\ ,$$
$$z^+ \leq I(T(p,w^+),r^+)$$
$$\Longleftrightarrow z^- \leq \min\{I(w^-,r^-), I(w^+,r^+)\},$$
$$z^+ \leq \min\{I(T(p,w^+),r^+), I(w^-,r^+)\}$$
$$\Longleftrightarrow z \leq_L \mathcal{I}_{I,p}(w,r).$$

Hence $(\mathcal{T}_{T,p}, \mathcal{I}_{I,p})$ constructs an interval-valued adjoint couple.

(iii) Owing to that $(\mathcal{T}_{T,p}, \mathcal{I}_{I,p})$ is an interval-valued adjoint couple, we have $\mathcal{T}_{T,p}(w,\gamma) \leq_L z$ if and only if $\gamma \leq_L \mathcal{I}_{I,p}(w,z)$ (in which $\gamma \in L$). Hence one has $\sup\{\gamma \in L \mid \mathcal{T}_{T,p}(w,\gamma) \leq_L z\} = \sup\{\gamma \in L \mid \gamma \leq_L \mathcal{I}_{I,p}(w,z)\} = \mathcal{I}_{I,p}(w,z)$. ■

When $p = 0$, then (10) is changed into $\mathcal{T}_{T,p}(w,z) = [T(w^-,z^-), \max\{T(w^-,z^+), T(w^+,z^-), T(T(0,w^+),z^+)\}] = [T(w^-,z^-), \max\{T(w^-,z^+), T(w^+,z^-)\}]$, which is consistent with the pessimistic t-norm with representative T (see Definition 2.14).

In Proposition 2.3, $\mathcal{T}_{T,p}$ is adjoint to $\mathcal{I}_{I,p}$, then such adjoint condition is corresponding to the adjunction property of Definition 2.10. In addition, $\mathcal{I}_{I,p}, \mathcal{T}_{T,p}$ match $\rightarrow, \otimes$ of Definition 2.10. For this reason, the truth-values of this work are under the meaning of complete residuated lattice.

III. The Basic GSI Method

On the strength of the idea of the basic GSI method and the environment of complete residuated lattice and granular computing, we set up the following principle:

Basic GSI principle: The result $\mathcal{Q}^\diamond$ of FMP (2) is the smallest interval-valued fuzzy set (in $<\Phi(Z), \leq_\Phi>$) making

$$\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{Q}^\diamond(z))) \tag{11}$$

gain its maximum for any $w \in W, z \in Z$, in which $\mathcal{I}_1, \mathcal{I}_2$ are two interval-valued implications.

The symmetric implicational principle in [14] is in view of the generic fuzzy sets, while the GSI principle is acquired from the interval-valued fuzzy sets. Hence, in the GSI principle, the computing procedure and the obtained result $\mathcal{Q}^\diamond$ of FMP (2) are all interrelated to interval-valued fuzzy sets. These are better than the corresponding cases from generic fuzzy sets. Draw a conclusion, such symmetric implicational principle for FMP ameliorates the previous one presented in [14].

Definition III.1: Let $\mathcal{P}, \mathcal{P}^\diamond \in \Phi(W)$, $\mathcal{Q} \in \Phi(Z)$, if $\mathcal{Q}^\diamond$ (in $<\Phi(Z), \leq_\Phi>$) makes (11) be maximized for any $w \in W$, $z \in Z$. Then $\mathcal{Q}^\diamond$ is referred to as a GSI solution.

Definition III.2: Assume that $\mathcal{P}, \mathcal{P}^\diamond \in \Phi(W)$, $\mathcal{Q} \in \Phi(Z)$, and that nonempty set $\mathbb{B}$ is the set of total GSI solutions, and that $\mathcal{O}^\diamond$ is the infimum of $\mathbb{B}$. Then $\mathcal{O}^\diamond$ goes by the name of a GInf-solution. In addition, if $\mathcal{O}^\diamond$ is the minimum of $\mathbb{B}$, then $\mathcal{O}^\diamond$ is referred to as a GMin-solution.

We adopt $\mathbf{M}(w,z)$ to represent the maximum of (11) at (w,z) for FMP.

Proposition III.1: (i) $\mathbf{M}(w,z) = \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), 1_L)) \equiv 1_L$ ($w \in W, z \in Z$).

(ii) $\mathcal{I}_1, \mathcal{I}_2$ take $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ in turn, where I_1, I_2 are implications on [0,1] meeting (PR5), and $p \in [0,1]$, then $\mathbf{M}(w,z) \equiv 1_L$ ($w \in W, z \in Z$).

Proof: (i) As $\mathcal{I}_1, \mathcal{I}_2$ satisfy (P2) (from Proposition 2.2), one has $\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{Q}^\diamond(z))) \leq_L \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), 1_L))$. We select $\mathcal{O}^\diamond = 1_L$, then (11) is equal to $\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), 1_L))$ ($w \in W, z \in Z$). Hence we have $\mathbf{M}(w,z) = \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), 1_L))$. Take into consideration that $\mathcal{I}_1, \mathcal{I}_2$ meet (P4) (from Proposition 2.2). Then $\mathbf{M}(w,z) = \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), 1_L) = 1_L$. (ii) In consideration of Proposition 2.2, $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ meet (P2), (P4), then it implies that $\mathbf{M}(w,z) \equiv 1_L$ ($w \in W, z \in Z$). ■

Proposition 3.2 can be acquired in a similar manner.

Proposition III.2: (i) If $\mathcal{Q}_1$ is a GSI solution, and $\mathcal{Q}_1 \leq_F \mathcal{Q}_2$ (in which $\mathcal{Q}_1, \mathcal{Q}_2 \in <\Phi(Z), \leq_\Phi>$), then $\mathcal{Q}_2$ is a GSI solution.

(ii) Let $\mathcal{I}_1, \mathcal{I}_2$ be $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ in turn, in which I_1, I_2 are implications on [0,1] meeting (PR5), and $p \in [0,1]$. $\mathcal{Q}_1$ is a GSI solution, and $\mathcal{Q}_1 \leq_F \mathcal{Q}_2$ (in which $\mathcal{Q}_1, \mathcal{Q}_2 \in <\Phi(Z), \leq_\Phi>$), then $\mathcal{Q}_2$ is a GSI solution.

Remark III.1: In accordance with Proposition 3.2, for any GSI solution $\mathcal{Q}^\diamond$, each $\mathcal{Q}_1$ in $<\Phi(Z), \leq_\Phi>$ such that $\mathcal{Q}^\diamond \leq_F \mathcal{Q}_1$, shall be also a GSI solution. Accordingly there exist many GSI solutions, which consist of $\mathcal{Q}_2^\diamond(z) \equiv 1_L$ ($z \in Z$). $\mathcal{Q}_2^\diamond$ is an extraordinary solution, due to that (11) all the time gets its maximum regardless of what $\mathcal{P}, \mathcal{P}^\diamond, \mathcal{Q}$ are chosen. For this reason, if the ideal GSI solution exists, then it should be the smallest one (or the infimum) of $\mathbb{B}$.

Proposition III.3: Suppose that $\mathcal{I}_1, \mathcal{I}_2$ satisfy

(P5) $\mathcal{I}(w,z)$ is right-continuous w.r.t z ($w, z \in L$),

then the GInf-solution is the GMin-solution.

Proof: It is manifested that the GInf-solution $\mathcal{O}^\diamond = \inf \mathbb{B}$, in which $\mathbb{B} = \{\mathcal{Q}_1^\diamond \in \Phi(Z) \mid \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{Q}_1^\diamond(z))) = \mathbf{M}(w,z),\ w \in W, z \in Z\}$.

We develop the proof by contradiction. Assume that $\mathcal{O}^\diamond \notin \mathbb{B}$, then there are $\mathcal{O}_1, \mathcal{O}_2, \cdots$ in $\mathbb{B}$ letting $\lim_{i\to\infty} \mathcal{O}_i(z) = \mathcal{O}^\diamond(z), \quad z \in Z$. In consideration of $\mathcal{O}_1, \mathcal{O}_2, \cdots \in \mathbb{B}$, one has $\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}_i(z))) = \mathbf{M}(w, z)$. $(w \in W,\ z \in Z,\ i = 1, 2, \cdots)$.

In virtue of $\mathcal{O}^\diamond = \inf \mathbb{B}$, we get $\mathcal{O}_i(z) \geq_L \mathcal{O}^\diamond(z)$ $(z \in Z,\ i = 1, 2, \cdots)$, and thus we have from $\lim_{i\to\infty} \mathcal{O}_i(z) = \mathcal{O}^\diamond(z)$ that $\mathcal{O}^\diamond(z)$ is the right limit of $\{\mathcal{O}_i(z) \mid i = 1, 2, \cdots\}$ $(z \in Z)$. Since $\mathcal{I}_1, \mathcal{I}_2$ meet (P2) and (P5), it means that $\lim_{i\to\infty}\{\mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}_i(z))\} = \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}^\diamond(z))$, and that $\mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}_i(z)) \geq_L \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}^\diamond(z))$ $(i = 1, 2, \cdots)$. Then $\mathbf{M}(w, z) = \lim_{i\to\infty}\{\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}_i(z)))\}$ $= \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}^\diamond(z)))$ $(w \in W, z \in Z)$. Thus $\mathcal{O}^\diamond \in \mathbb{B}$, which reflects a contradiction.

For this reason, $\mathcal{O}^\diamond \in \mathbb{B}$ and hence $\mathcal{O}^\diamond$ is the minimum of $\mathbb{B}$, which implies that the GInf-solution $\mathcal{O}^\diamond$ is the GMin-solution. ■

Theorem III.1: If I_1, I_2 are implications on [0,1] meeting (PR5), and $\mathcal{T}^{(1)}_{T,p}, \mathcal{T}^{(2)}_{T,p}$ are the functions adjoint to $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ in turn (where $p \in [0, 1]$), then the GMin-solution $\mathcal{Q}^\diamond$ can be calculated as below ($z \in Z$):

$$\mathcal{Q}^\diamond(z) = \sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)),\ 1_L))\}. \tag{12}$$

Proof: As I_1, I_2 are implications on [0,1] meeting (PR5), it follows from Proposition 2.3 that there are $\mathcal{T}^{(1)}_{T,p}, \mathcal{T}^{(2)}_{T,p}$ which are adjoint to $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ in turn. From (12), one has that $\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)),\ 1_L)) \leq_L \mathcal{Q}^\diamond(z)$ $(w \in W, z \in Z)$.

Taking into consideration that $(\mathcal{T}^{(2)}_{T,p}, \mathcal{I}_{I_2,p}), (\mathcal{T}^{(1)}_{T,p}, \mathcal{I}_{I_1,p})$ are two interval-valued adjoint couples, one acquires from (9) that $\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)),\ 1_L) \leq_L \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w), \mathcal{Q}^\diamond(z))$, and that $1_L \leq_L \mathcal{I}_{I_2,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w), \mathcal{Q}^\diamond(z)))$ $(w \in W, z \in Z)$, which means that $1_L = \mathcal{I}_{I_2,p}(\mathcal{I}_{I_1,p}$ $(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w), \mathcal{Q}^\diamond(z)))$.

As I_1, I_2 satisfy (PR5), we get from Proposition 3.1 that $\mathbf{M}(w, z) \equiv 1_L$ $(w \in W, z \in Z)$. Consequently, $\mathcal{Q}^\diamond$ denoted by (12) let (11) be maximized $(w \in W, z \in Z)$, that is, $\mathcal{Q}^\diamond \in \mathbb{B}$.

Afterwards we validate that $\mathcal{Q}^\diamond$ represented as (12) is the minimum of entire GSI solutions.

Assume that $\mathcal{Q}_1$ is any GSI solution, viz., $\mathcal{Q}_1 \in \mathbb{B}$, then it implies $1_L = \mathcal{I}_{I_2,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w), \mathcal{Q}_1(z)))$ $(w \in W, z \in Z)$. As $(\mathcal{T}^{(2)}_{T,p}, \mathcal{I}_{I_2,p}), (\mathcal{T}^{(1)}_{T,p}, \mathcal{I}_{I_1,p})$ are two interval-valued adjoint couples, we find $1_L \leq_L \mathcal{I}_{I_2,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)), < ?brk? > \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w), \mathcal{Q}_1(z)))$, and $\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)),\ 1_L) \leq_L \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w), \mathcal{Q}_1(z))$, and $\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)),\ 1_L)) \leq_L \mathcal{Q}_1(z)$ $(w \in W, z \in Z)$. Accordingly, $\mathcal{Q}_1(z)$ is an upper bound of $\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)),\ 1_L)) \mid w \in W\}, z \in Z$. Therefore one has from (12) that $\mathcal{Q}^\diamond \leq_\Phi \mathcal{Q}_1$. As a consequence, $\mathcal{Q}^\diamond$ indicated by (12) is the minimum of $\mathbb{B}$.

In a word, $\mathcal{Q}^\diamond$ indicated by (12) is the GMin-solution. ■

It is effortless to prove Lemma 3.1.

Lemma III.1: Let I be an implication on [0,1] meeting (PR5), (PR7), (PR9), and $p \in [0, 1]$, then the function $\mathcal{T}_{T,p}$ adjoint to $\mathcal{I}_{I,p}$ meets

(P6) $\mathcal{T}_{T,p}(w, 1_L) = w \quad (w \in L)$.

Theorem III.2: If I_1 is an (S,N)-implication meeting (PR5) or an R-implication, and I_2 is an R-implication, and $\mathcal{T}^{(1)}_{T,p}, \mathcal{T}^{(2)}_{T,p}$ are respectively the functions adjoint to $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ (where $p \in [0, 1]$), then the GMin-solution $\mathcal{Q}^\diamond$ can be computed as below ($z \in Z$):

$$\mathcal{Q}^\diamond(z) = \sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)))\}. \tag{13}$$

Proof: (i) Assume that I_1, I_2 are two R-implications. In accordance with Lemma 2.2, we gain that I_1, I_2 satisfy (PR5), (PR7), (PR9). Thus one has from Lemma 3.1 that $\mathcal{T}^{(2)}_{T,p}$ meets (P6). For this reason, we acquire from Theorem 3.1 that the GMin-solution $\mathcal{Q}^\diamond$ is $\mathcal{Q}^\diamond(z) = \sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)), 1_L))\} = \sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{Q}(z)))\},\ z \in Z$.

(ii) Assume that I_1 is an (S,N)-implication meeting (PR5), and I_2 is an R-implication. From the property of I_2 we find that $\mathcal{T}^{(2)}_{T,p}$ satisfies (P6). Then we can analogously gain that the GMin-solution $\mathcal{Q}^\diamond$ can be represented by (13). ■

Theorem III.3: (i) If I_1 adopts an (S,N)-implication meeting (PR5) or an R-implication, and I_2 takes an (S,N)-implication satisfying (PR5), and $\mathcal{T}^{(1)}_{T,p}, \mathcal{T}^{(2)}_{T,p}$ are the functions adjoint to $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ in turn (where $p \in [0, 1]$), then the GMin-solution $\mathcal{Q}^\diamond$ can be signified by (12).

(ii) Particularly, if I_2 also meets (PR9), then the GMin-solution $\mathcal{Q}^\diamond$ can be calculated as (13).

Proof: (i) As I_1 adopts an (S,N)-implication meeting (PR5) or an R-implication and I_2 utilizes an (S,N)-implication meeting (PR5), it follows from Theorem 3.1 that the GMin-solution $\mathcal{Q}^\diamond$ can be calculated as (12).

(ii) We have from Proposition 2.1 that I_2 meets (PR7). Hence (PR5), (PR7), (PR9) hold for I_2. As a consequence one gets from Lemma 3.1 that $\mathcal{T}^{(2)}_{T,p}$ meets (P6), and then the GMin-solution $\mathcal{Q}^\diamond$ denoted by (12) can be transformed into (13). ■

We introduce some R-implications including I_L, I_F, I_{GG}, I_{GD}, I_E (adjoint to the t-norm of Einstein product), I_Y (adjoint to the t-norm of Yager), and (S, N)-implications incorporating I_R, I_{KD}, I_{MK}, I_{TD}, I_D [15]. Note that I_L, I_F are also (S, N)-implications. The formulae of these implications are as follows:

$$I_L(w,z) = \begin{cases} 1 & \text{if } w \leq z \\ 1 - w + z & \text{if } w > z \end{cases},$$

$$I_F(w,z) = \begin{cases} 1 & \text{if } w \leq z \\ (1 - w) \vee z & \text{if } w > z \end{cases},$$

$$I_{GG}(w,z) = \begin{cases} 1 & \text{if } w \leq z \\ z/w & \text{if } w > z \end{cases},$$

$$I_{GD}(w,z) = \begin{cases} 1 & \text{if } w \leq z \\ z & \text{if } w > z \end{cases},$$

$$I_E(w,z) = \begin{cases} 1 & \text{if } w \leq z \\ (2z - wz)/(w + z - wz) & \text{if } w > z \end{cases},$$

$$I_Y(w,z) = \begin{cases} 1 & \text{if } w \leq z \\ 1 - (\sqrt{1-z} - \sqrt{1-w}^2 & \text{if } w > z \end{cases},$$

$I_R(w,z) = 1 - w + wz$, $I_{KD}(w,z) = (1 - w) \vee z$, $I_{MK}(w,z) = (1 - w^2) \vee z$,

$$I_{TD}(w,z) = \begin{cases} 1 & \text{if } w < 1 \\ z & \text{if } w = 1 \end{cases},\ I_D(w,z) = \begin{cases} 1 & \text{if } w = 0 \\ z & \text{if } w > 0 \end{cases}.$$

It is effortless to know that (PR5) works for the (S, N)-implications $I_D, I_{TD}, I_{KD}, I_R, I_F, I_L, I_{MK}$, and that (PR9) holds for the (S, N)-implications I_F, I_L. Consequently, Proposition 3.4 is deduced from Theorem 3.2 and Theorem 3.3.

Proposition III.4: (i) If $\rightarrow_1, \rightarrow_2 \in \{I_R, I_{MK}, I_E, I_{GG}, I_F, I_L, I_D, I_{GD}, I_Y, I_{KD}, I_{TD}\}$, *and* $\mathcal{T}^{(1)}_{T,p}$, $\mathcal{T}^{(2)}_{T,p}$ *are respectively the functions adjoint to* $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ *(thereinto* $p \in [0,1]$*), then the GMin-solution* $\mathcal{Q}^\diamond$ *is counted by (12).*

(ii) If $\rightarrow_1 \in \{I_L, I_Y, I_{GD}, I_{MK}, I_D, I_F, I_E, I_{GG}, I_R, I_{KD}, I_{TD}\}$ *and* $\rightarrow_2 \in \{I_{GG}, I_F, I_L, I_{GD}, I_E, I_Y\}$, *and* $\mathcal{T}^{(1)}_{T,p}, \mathcal{T}^{(2)}_{T,p}$ *are respectively the functions adjoint to* $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ *(thereinto* $p \in [0,1]$*), then the GMin-solution* $\mathcal{Q}^\diamond$ *is calculated by (13).*

When the input is changed from one fuzzy set to multiple ones, the status of multiple-input and single-output (MISO) is considered, which is more practically relevant. Here we show the status of double-input and single-output (DISO), which can be simply extended into MISO. The FMP problem of DISO in the interval-valued fuzzy environment is shown as below:

$$\text{From } \mathcal{P}, \mathcal{Q} \Longrightarrow \mathcal{O}, \text{ and inputs} \mathcal{P}^\diamond, \mathcal{Q}^\diamond, \text{Gain } \mathcal{O}^\diamond, \tag{14}$$

where $\mathcal{P}, \mathcal{P}^\diamond \in \Phi(W)$, $\mathcal{Q}, \mathcal{Q}^\diamond \in \Phi(Z)$, $\mathcal{O}, \mathcal{O}^\diamond \in \Phi(Y)$.

In [34], Jayaram came up with the hierarchical CRI method, which exhibited good properties from the viewpoints of computational efficiency, storage efficiency, associative inferencing and order independence. Afterwards, here we also employ the hierarchical granular structure for the basic GSI method, referred here as the hierarchical basic GSI method.

The hierarchical basic GSI method proceeds in two steps where we adopt the GMin-solution shown by (12) (see Theorem 3.1 and Theorem 3.3):

(i) We put to use the basic GSI method with $\mathcal{Q} \Longrightarrow \mathcal{O}$ and $\mathcal{Q}^\diamond$. Then by virtue of (12), we can acquire the in-between solution $\mathcal{O}_1(y) = \sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}^\diamond(z), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{Q}(z), \mathcal{O}(y)),\ 1_L))\}$ $(y \in Y)$. Thereinto $\mathcal{T}^{(1)}_{T,p}, \mathcal{T}^{(2)}_{T,p}$ are the mappings adjoint to $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$.

(ii) Then we utilize the basic GSI method with $\mathcal{P} \Longrightarrow \mathcal{O}_1$ and $\mathcal{P}^\diamond$. From (12), we gain the last output $\mathcal{O}^\diamond$ $(y \in Y)$:

$$\mathcal{O}^\diamond(y) = \sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}^\diamond(z), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{Q}(z), \mathcal{O}(y)),\ 1_L))\}),\ 1_L))\}. \tag{15}$$

Analogously, for the GMin-solution $\mathcal{Q}^\diamond$ shown by (13) (see Theorem 3.2 and Theorem 3.3), we can gain the hierarchical solution as below $(y \in Y)$:

$$\mathcal{O}^\diamond(y) = \sup_{w\in W}\left\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{I}_{I_1,p}(\mathcal{P}(w), \sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p} (\mathcal{Q}^\diamond(z), \mathcal{I}_{I_1,p}(\mathcal{Q}(z), \mathcal{O}(y)))\}))\right\}. \tag{16}$$

IV. The $\zeta(w,z)$-GSI Method

From the mechanism of the $\zeta(w,z)$-GSI method, the complete residuated lattice and granular computing, we offer the following finding:

$\zeta(w,z)$**-GSI principle:** The outcome $\mathcal{Q}^\diamond$ of FMP (2) is the smallest one (in $<\Phi(Z), \leq_\Phi>$) letting

$$\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{Q}^\diamond(z))) \geq_L \zeta(w,z) \tag{17}$$

work for any $w \in W, z \in Z$. Thereinto $\mathcal{I}_1, \mathcal{I}_2$ are two interval-valued implications, and $\zeta(w,z) = [\zeta^-(w,z), \zeta^+(w,z)]$ where ζ^-, ζ^+ are two functions from $W \times Z$ to [0,1], and $\zeta^-(w,z) \leq \zeta^+(w,z)$ for any $w \in W, z \in Z$.

In like manner, such symmetric implicational principle for FMP ameliorates the previous one afforded in [15].

Definition IV.1: Let $\mathcal{P}, \mathcal{P}^\diamond \in \Phi(W)$, $\mathcal{Q} \in \Phi(Z)$, *if* $\mathcal{Q}^\diamond$ *(in* $<\Phi(Z), \leq_\Phi>$*) makes (17) hold for any* $w \in W$, $z \in Z$. *Then* $\mathcal{Q}^\diamond$ *is referred to as a* $\zeta(w,z)$*-GSI solution.*

Definition IV.2: Assume that $\mathcal{P}, \mathcal{P}^\diamond \in \Phi(W)$, $\mathcal{Q} \in \Phi(Z)$, *and that nonempty set* $\mathbb{B}_{\zeta(w,z)}$ *is the set of total* $\zeta(w,z)$*-GSI solutions, and that* $\mathcal{O}^\diamond$ *is the infimum of* $\mathbb{B}_{\zeta(w,z)}$. *Then* $\mathcal{O}^\diamond$ *goes by the name of a* $\zeta(w,z)$*-GInf-solution. In addition, if* $\mathcal{O}^\diamond$ *is also the minimum of* $\mathbb{B}_{\zeta(w,z)}$, *then* $\mathcal{O}^\diamond$ *is referred to as a* $\zeta(w,z)$*-GMin-solution.*

Analogous to Proposition 3.2, Proposition 4.1 is offered.

Proposition IV.1: (i) If $\mathcal{Q}_1$ *is a* $\zeta(w,z)$*-GSI solution, and* $\mathcal{Q}_1 \leq_F \mathcal{Q}_2$ *(in which* $\mathcal{Q}_1, \mathcal{Q}_2 \in <\Phi(Z), \leq_\Phi>$*), then* $\mathcal{Q}_2$ *is a* $\zeta(w,z)$*-GSI solution.*

(ii) Let $\mathcal{I}_1, \mathcal{I}_2$ *be* $\mathcal{I}_{I_1,p}, \mathcal{I}_{I_2,p}$ *in turn, in which* I_1, I_2 *are implications on [0,1] meeting (PR5), and* $p \in [0,1]$. $\mathcal{Q}_1$ *is a* $\zeta(w,z)$*-GSI solution, and* $\mathcal{Q}_1 \leq_F \mathcal{Q}_2$ *(in which* $\mathcal{Q}_1, \mathcal{Q}_2 \in <\Phi(Z), \leq_\Phi>$*), then* $\mathcal{Q}_2$ *is a* $\zeta(w,z)$*-GSI solution.*

Remark IV.1: In consideration of Proposition 4.1, for any $\zeta(w,z)$*-GSI solution* $\mathcal{Q}^\diamond$, *each* $\mathcal{Q}_1$ *in* $<\Phi(Z), \leq_\Phi>$ *making* $\mathcal{Q}^\diamond \leq_F \mathcal{Q}_1$, *shall be also a* $\zeta(w,z)$*-GSI solution. Accordingly there exist many* $\zeta(w,z)$*-GSI solutions, which consist of* $\mathcal{Q}^\diamond_2(z) \equiv 1_L$ $(z \in Z)$. *Here* $\mathcal{Q}^\diamond_2$ *is an extraordinary solution, due to that (11) all the time takes its maximum regardless of what* $\mathcal{P}, \mathcal{P}^\diamond, \mathcal{Q}$ *are chosen. For this reason, if the ideal* $\zeta(w,z)$*-GSI solution exists, then it should be the smallest one (or the infimum) in* $\mathbb{B}_{\zeta(w,z)}$.

Proposition IV.2: Assume that $\mathcal{I}_1, \mathcal{I}_2$ *meet (P5), then the* $\zeta(w,z)$*-GInf-solution is the* $\zeta(w,z)$*-GMin-solution.*

Proof: It is manifest that the $\zeta(w,z)$-GInf-solution $\mathcal{O}^\diamond = \inf \mathbb{B}_{\zeta(w,z)}$, in which $\mathbb{B}_{\zeta(w,z)} = \{\mathcal{Q}^\diamond_1 \in \Phi(Z) \mid \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{Q}^\diamond_1(z))) \geq_L \zeta(w,z),\ w \in W, z \in Z\}$.

We adopt the proof by contradiction. Assume that $\mathcal{O}^\diamond \notin \mathbb{B}_{\zeta(w,z)}$, then there are $\mathcal{O}_1, \mathcal{O}_2, \cdots$ in $\mathbb{B}_{\zeta(w,z)}$ letting $\lim_{i\to\infty} \mathcal{O}_i(z) = \mathcal{O}^\diamond(z)$, $z \in Z$. In consideration of $\mathcal{O}_1, \mathcal{O}_2, \cdots \in \mathbb{B}_{\zeta(w,z)}$, we get $\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w), \mathcal{Q}(z)), \mathcal{I}_1(\mathcal{P}^\diamond(w), \mathcal{O}_i(z))) \geq_L \zeta(w,z)$. $(w \in W,\ z \in Z,\ i = 1, 2, \cdots)$.

In virtue of $\mathcal{O}^\diamond = \inf \mathbb{B}_{\zeta(w,z)}$, we have $\mathcal{O}_i(z) \geq_L \mathcal{O}^\diamond(z)$ $(z \in Z,\ i = 1, 2, \cdots)$, and hence it follows from $\lim_{i\to\infty} \mathcal{O}_i(z) = \mathcal{O}^\diamond(z)$ that $\mathcal{O}^\diamond(z)$ is the right limit of $\{\mathcal{O}_i(z) \mid i = 1, 2, \cdots\}$ $(z \in Z)$. Because $\mathcal{I}_1, \mathcal{I}_2$ satisfy (P2) and (P5), we find that $\lim_{i\to\infty}\{\mathcal{I}_1(\mathcal{P}^\diamond(w),\mathcal{O}_i(z))\}=\mathcal{I}_1(\mathcal{P}^\diamond(w),\mathcal{O}^\diamond(z))$, and that $\mathcal{I}_1(\mathcal{P}^\diamond(w),\mathcal{O}_i(z))\geq_L \mathcal{I}_1(\mathcal{P}^\diamond(w),\mathcal{Q}^\diamond(z))$ $(w\in W, z\in Z, i=1,2,\cdots)$. Then one has $\zeta(w,z)\leq_L \lim_{i\to\infty}\{\mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w),\mathcal{Q}(z)),\mathcal{I}_1(\mathcal{P}^\diamond(w),\mathcal{O}_i(z)))\} = \mathcal{I}_2(\mathcal{I}_1(\mathcal{P}(w),\mathcal{Q}(z)),\mathcal{I}_1(\mathcal{P}^\diamond(w),\mathcal{O}^\diamond(z)))$ $(w\in W, z\in Z)$. So $\mathcal{O}^\diamond\in\mathbb{B}_{\zeta(w,z)}$, which forms a contradiction.

For this reason, $\mathcal{O}^\diamond\in\mathbb{B}_{\zeta(w,z)}$ and hence $\mathcal{O}^\diamond$ is the minimum of $\mathbb{B}_{\zeta(w,z)}$, which means that the $\zeta(w,z)$-GInf-solution $\mathcal{O}^\diamond$ is the $\zeta(w,z)$-GMin-solution. ■

Theorem IV.1: If I_1, I_2 are implications on [0,1] meeting (PR5), and $\mathcal{T}^{(1)}_{T,p},\mathcal{T}^{(2)}_{T,p}$ are the function adjoint to $\mathcal{I}_{I_1,p},\mathcal{I}_{I_2,p}$ in turn (where $p\in[0,1]$), then the $\zeta(w,z)$-GMin-solution $\mathcal{Q}^\diamond(z)$ can be calculated as below ($z\in Z$):

$$\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\ \zeta(w,z)))\}. \tag{18}$$

Proof: Owing to that I_1, I_2 are implications on [0,1] meeting (PR5), one has from Proposition 2.3 that there exist $\mathcal{T}^{(1)}_{T,p},\mathcal{T}^{(2)}_{T,p}$ which are adjoint to $\mathcal{I}_{I_1,p},\mathcal{I}_{I_2,p}$ in turn. By virtue of (18), we gain that $\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\ \zeta(w,z)))\leq_L \mathcal{Q}^\diamond(z)$ $(w\in W, z\in Z)$. Take into consideration that $(\mathcal{T}^{(2)}_{T,p},\mathcal{I}_{I_2,p}),(\mathcal{T}^{(1)}_{T,p},\mathcal{I}_{I_1,p})$ are two interval-valued adjoint couples, which implies that $\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\ \zeta(w,z))\leq_L \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w),\mathcal{Q}^\diamond(z))$, and that $\zeta(w,z)\leq_L \mathcal{I}_{I_2,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w),\mathcal{Q}^\diamond(z)))$ $(w\in W, z\in Z)$. As a consequence, $\mathcal{Q}^\diamond$ represented as (18) entails (17) hold for any $w\in W, z\in Z$, i.e., $\mathcal{Q}^\diamond\in\mathbb{B}_{\zeta(w,z)}$.

In addition, we validate that $\mathcal{Q}^\diamond$ indicated by (18) is the minimum of entire $\zeta(w,z)$-GSI solutions. Assume that $\mathcal{O}$ is any $\zeta(w,z)$-GSI solution, viz., $\mathcal{O}\in\mathbb{B}_{\zeta(w,z)}$, then one has $\zeta(w,z)\leq_L \mathcal{I}_{I_2,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w),\mathcal{O}(z)))$ $(w\in W, z\in Z)$. With a view to that $(\mathcal{T}^{(2)}_{T,p},\mathcal{I}_{I_2,p}),(\mathcal{T}^{(1)}_{T,p},\mathcal{I}_{I_1,p})$ are two interval-valued adjoint couples, we acquire $\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\ \zeta(w,z))\leq_L \mathcal{I}_{I_1,p}(\mathcal{P}^\diamond(w),\mathcal{O}(z))$, and $\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\ \zeta(w,z)))\leq_L \mathcal{O}(z)$ $(w\in W,\ z\in Z)$. As a consequence, $\mathcal{O}(z)$ is an upper bound of $\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\ \zeta(w,z)))\mid w\in W\}$, $z\in Z$. Hence we have from (18) that $\mathcal{Q}^\diamond\leq_\Phi \mathcal{O}$. Accordingly, $\mathcal{Q}^\diamond$ denoted by (18) is the minimum of $\mathbb{B}_{\zeta(w,z)}$.

Thus $\mathcal{Q}^\diamond$ represented by (18) is the $\zeta(w,z)$-GMin-solution. ■

Theorem IV.2: If I_1 is an (S,N)-implication meeting (PR5) or an R-implication, and I_2 is an (S,N)-implication meeting (PR5) or an R-implication, and $\mathcal{T}^{(1)}_{T,p},\mathcal{T}^{(2)}_{T,p}$ are the functions adjoint to $\mathcal{I}_{I_1,p},\mathcal{I}_{I_2,p}$ in turn (where $p\in[0,1]$), then the $\zeta(w,z)$-GMin-solution $\mathcal{Q}^\diamond$ is calculated as (18).

Proof: In consideration of Lemma 2.2, an R-implication meets (PR5), too. Hence I_1 and I_2 both meet (PR5). As a consequence, one has from Theorem 4.1 that the $\zeta(w,z)$-GMin-solution $\mathcal{Q}^\diamond$ is computed as (18). ■

Theorem 4.1 and Theorem 4.2 can infer Proposition 4.3.

Proposition IV.3: If $\rightarrow_1,\rightarrow_2\in\{I_E,I_Y,I_{KD},I_{MK},I_D,I_R,I_L,I_{GD},I_{GG},I_F,I_{TD}\}$, and $\mathcal{T}^{(1)}_{T,p},\mathcal{T}^{(2)}_{T,p}$ are the functions adjoint to $\mathcal{I}_{I_1,p},\mathcal{I}_{I_2,p}$ (thereinto $p\in[0,1]$), then the $\zeta(w,z)$-GMin-solution $\mathcal{Q}^\diamond$ can be denoted by (18).

For the MISO status (14), it is analogous to the hierarchical basic GSI method that we can come up with the hierarchical $\zeta(w,z)$-GSI method. To be specific, it can be deduced by two steps in which we adopt the $\zeta(w,z)$-GMin-solution represented by (18) (paying respects to Theorem 4.1 and Theorem 4.2):

(i) We make use of the $\zeta(w,z)$-GSI method with $\mathcal{Q}\Longrightarrow\mathcal{O}$ and $\mathcal{Q}^\diamond$. Then in consideration of (18), we can gain the in-between solution $\mathcal{O}_1(y)=\sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}^\diamond(z),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)),\quad \zeta(w,z)))\}$ $(y\in Y)$ Here $\mathcal{T}^{(1)}_{T,p},\mathcal{T}^{(2)}_{T,p}$ are the functions adjoint to $\mathcal{I}_{I_1,p},\mathcal{I}_{I_2,p}$.

(ii) We adopt the $\zeta(w,z)$-GSI method with $\mathcal{P}\Longrightarrow\mathcal{O}_1$ and $\mathcal{P}^\diamond$. From (18), we acquire the last output $\mathcal{O}^\diamond(y)=\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{O}_1(y)),\quad \zeta(w,z)))\}$ $(y\in Y)$.

For the sake of controlling every step, we can utilize different $\zeta(w,z)$. For the DISO status, we can adopt $\zeta_1(w,z),\zeta_2(w,z)$, then $\mathcal{O}^\diamond(y)$ is $(y\in Y)$:

$$\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}^\diamond(z), \mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)),\ \zeta_2(w,z)))\}),\ \zeta_1(w,z)))\}. \tag{19}$$

V. The Properties of the GSI Method

A. The Reversibility Properties of the Basic GSI Method

For any fuzzy inference strategy, there is no completely accepted criterion using which one can estimate its quality. However, its reversibility property reflects the compatibility with classic logic, which is deemed as a requirement.

Definition V.1: In allusion to a method to figure out FMP (2), if $\mathcal{P}^\diamond=\mathcal{P}$ results in $\mathcal{Q}^\diamond=\mathcal{Q}$ if condition CN works, then this method is said to be CN-reversible.

Definition V.2: In allusion to a method to figure out FMP (14) for the DISO status, if $\mathcal{P}^\diamond=\mathcal{P}$ and $\mathcal{Q}^\diamond=\mathcal{Q}$ result in $\mathcal{O}^\diamond=\mathcal{O}$ if condition CN is effective, then it is said to be CN-reversible.

It is effortless to gain Lemma 5.1.

Lemma V.1: Assume that the implication I on [0,1] meets (PR5) and (PR7), and that T is the function adjoint to I, and that $\mathcal{T}_{T,p}$ is the function adjoint to $\mathcal{I}_{I,p}$ (where $p\in[0,1]$), then one has: (i) $z\leq I(w,z)$, $T(w,z)\leq z$ $(w,z\in[0,1])$, (ii) $T(1,w)=w$ $(w\in[0,1])$, (iii) $\mathcal{I}_{I,p}(1_L,w)=w$ $(w\in L)$, (iv) $\mathcal{T}_{T,p}(1_L,w)=w$ $(w\in L)$, (v) $\mathcal{T}_{T,p}$ is increasing in its two variables. In addition, if (PR9) also works for I, then we acquire: (vi) $T(w,1)=w$ $(w\in[0,1])$.

Theorem V.1: If the implication I_1 meets (PR5), (PR7), and the implication I_2 meets (PR5), (PR7), (PR9), then the basic GSI method represented by (12) owns reversibility property for the normal input (i.e. there is $w_0\in W$ letting $\mathcal{P}(w_0)=1_L$ be effective).

Proof: Since I_2 meets (PR5), (PR7), (PR9), from Lemma 3.1, one has that (P6) is effective for $\mathcal{T}^{(2)}_{T,p}$. From Definition 2.4, I_1, I_2 meets (PR1), (PR2), (PR3) and (PR4).

When $\mathcal{P}^\diamond=\mathcal{P}$, it follows from Theorem 3.1 that the GMin-solution can be expressed as $\mathcal{Q}^\diamond(z)=$

$\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\quad 1_L))\} = \sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)))\}\ (z\in Z)$.

Owing to that $\mathcal{P}$ is normal, there exists $w_0\in W$ making $\mathcal{P}(w_0)=1_L$. From Lemma 5.1, one has $\mathcal{Q}^\diamond(z)\geq_L \mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w_0),\mathcal{I}_{I_1,p}(\mathcal{P}(w_0),\mathcal{Q}(z)))=\mathcal{T}^{(1)}_{T,p}(1_L,\mathcal{I}_{I_1,p}(1_L,\mathcal{Q}(z)))=\mathcal{T}^{(1)}_{T,p}(1_L,\mathcal{Q}(z))=\mathcal{Q}(z)\ (z\in Z)$.

Subsequently, we show $\mathcal{Q}^\diamond(z)\leq_L\mathcal{Q}(z)\ (z\in Z)$. It is apparent that $\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z))\leq_L\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z))$ $(w\in W, z\in Z)$, then it follows from (9) that $\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)))\leq_L\mathcal{Q}(z)\ (w\in W, z\in Z)$. In consequence, $\mathcal{Q}^\diamond(z)\leq_L\mathcal{Q}(z)\ (z\in Z)$.

In a word, we obtain $\mathcal{Q}^\diamond=\mathcal{Q}$. ∎

Theorem V.2: If the implication I_1 meets (PR5), (PR7), and the implication I_2 meets (PR5), (PR7), (PR9), then the hierarchical basic GSI method computed by (15) has reversibility property for the normal input (viz. there are $w_0\in W$, $z_0\in Z$ letting $\mathcal{P}(w_0)=\mathcal{Q}(z_0)=1_L$ be effective).

Proof: Analogous to Theorem 5.1, one has that (P6) holds for $\mathcal{T}^{(2)}_{T,p}$. Let $\mathcal{P}^\diamond=\mathcal{P}$ and $\mathcal{Q}^\diamond=\mathcal{Q}$. The optimal solution (15) to the hierarchical basic GSI method can be transformed into $\mathcal{O}^\diamond(y)=\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}(z),\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)),\quad 1_L))\}),\quad 1_L))\}=\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{I}_{I_1,p}(\mathcal{P}(w),\sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}(z),\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)))\}))\}$ $(y\in Y)$.

Taking into consideration that $\mathcal{P}$ and $\mathcal{Q}$ are normal, there exist $w_0\in W$, $z_0\in Z$ such that $\mathcal{P}(w_0)=\mathcal{Q}(z_0)=1_L$. In virtue of Lemma 5.1, one has $\mathcal{O}^\diamond(y)\geq_L\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w_0),\mathcal{I}_{I_1,p}(\mathcal{P}(w_0),\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}(z_0),\mathcal{I}_{I_1,p}(\mathcal{Q}(z_0),\mathcal{O}(y)))))=\mathcal{T}^{(1)}_{T,p}(1_L,\mathcal{I}_{I_1,p}(1_L,\mathcal{T}^{(1)}_{T,p}(1_L,\mathcal{I}_{I_1,p}(1_L,\mathcal{O}(y)))))=\mathcal{T}^{(1)}_{T,p}(1_L,\mathcal{I}_{I_1,p}(1_L,\mathcal{O}(y)))=\mathcal{O}(y)\ (y\in Y)$.

It is apparent that $\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y))\leq_L\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y))$ $(z\in Z, y\in Y)$, then it follows from (9) that $\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}(z),\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)))\leq_L\mathcal{O}(y)\ (z\in Z, y\in Y)$. In a similar way, one has $\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{O}(y)))\leq_L\mathcal{O}(y)$ $(w\in W, y\in Y)$. As a result, $\mathcal{O}^\diamond(y)=\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{I}_{I_1,p}(\mathcal{P}(w),\sup_{z\in Z}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{Q}(z),\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)))\}))\}$ $\leq_L\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}(w),\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{O}(y)))\}\leq_L\mathcal{O}(y)$.

In conclusion, we gain $\mathcal{O}^\diamond=\mathcal{O}$. ∎

B. The Continuous Properties of the GSI Method

Focusing on (2) of fuzzy inference, when $P^\diamond$ is close to P, if the inference outcome $Q^\diamond$ is quite disparate from Q, then this reasoning method is hardly practical in the real world. Therefore, it is important that a small deviation in the input will not result in a substantial difference in the output (which is considered as the continuity issue of fuzzy inference).

A distance function d is known as a metric, if d satisfies $d(w,z)=d(z,w)$, $d(w,z)\geq 0$ (in which $d(w,z)=0$ iff $w=z$), and $d(w,r)\leq d(w,z)+d(z,r)$ for any points w,z,r. The concept of distance has been extended to fuzzy set. Assume that d is a distance between fuzzy sets, which constructs a metric.

We constantly adopt the distance between two intervals $a=[a^-,a^+]$ and $b=[b^-,b^+]$ in the form $d(w,z)=|w^- - z^-|\vee|w^+ - z^+|$.

The uniform metric d_{UN} is commonly-utilized, and we expand it for the granular sense (e.g., interval-valued fuzzy sets) as $d_{UN}(\mathcal{P}_1,\mathcal{P}_2)=\sup_{w\in W}\{|\mathcal{P}_1^-(w)-\mathcal{P}_2^-(w)|\vee|\mathcal{P}_1^+(w)-\mathcal{P}_2^+(w)|\}$ $(\mathcal{P}_1,\mathcal{P}_2\in\Phi(W))$.

Definition V.3: A fuzzy inference method for FMP (2) is a function $f:\Phi(W)\to\Phi(Z)$, viz., there exists an output $\mathcal{Q}^\diamond=f(\mathcal{P}^\diamond)\in\Phi(Z)$ for any input $\mathcal{P}^\diamond\in\Phi(W)$. (i) For any $\varepsilon>0$, if there exists $\delta>0$ making $d(f(\mathcal{P}_1),f(\mathcal{P}_2))<\varepsilon$ hold if $d(\mathcal{P}_1,\mathcal{P}_2)<\delta$ for any $\mathcal{P}_1,\mathcal{P}_2\in\Phi(W)$, then f is referred to as a uniformly continuous function in d. (ii) Aiming at any $\varepsilon>0$, if there exists $\delta>0$ making $d(f(\mathcal{P}_1),f(\mathcal{P}))<\varepsilon$ work if $d(\mathcal{P}_1,\mathcal{P})<\delta$ for any $\mathcal{P}_1\in\Phi(W)$, then f is known as a continuous function at $\mathcal{P}\in\Phi(W)$ in d.

Definition V.4: A fuzzy inference method for FMP (14) is a function $g:\Phi(W)*\Phi(Z)\to\Phi(Y)$, i.e., there is an output $\mathcal{O}^\diamond=g(\mathcal{P}^\diamond,\mathcal{Q}^\diamond)\in\Phi(Y)$ for inputs $\mathcal{P}^\diamond\in\Phi(W),\mathcal{Q}^\diamond\in\Phi(Z)$. (i) For any $\varepsilon>0$, if there exists $\delta>0$ letting $d(g(\mathcal{P}_1,\mathcal{Q}_1),g(\mathcal{P}_2,\mathcal{Q}_2))<\varepsilon$ hold if $d(\mathcal{P}_1,\mathcal{P}_2)<\delta$ and $d(\mathcal{Q}_1,\mathcal{Q}_2)<\delta$ for any $\mathcal{P}_1,\mathcal{P}_2\in\Phi(W)$ and $\mathcal{Q}_1,\mathcal{Q}_2\in\Phi(Z)$, then g is called to be uniformly continuous in d. (ii) Aiming at any $\varepsilon>0$, if there exists $\delta>0$ such that $d(g(\mathcal{P}_1,\mathcal{Q}_1),g(\mathcal{P},\mathcal{Q}))<\varepsilon$ if $d(\mathcal{P}_1,\mathcal{P})<\delta$ and $d(\mathcal{Q}_1,\mathcal{Q})<\delta$ for any $\mathcal{P}_1\in\Phi(W)$ and $\mathcal{Q}_1\in\Phi(Z)$, then g is said to be continuous at $\mathcal{P}\in\Phi(W)$ and $\mathcal{Q}\in\Phi(Z)$ in d.

Lemma V.2: If the t-norm T is continuous and $p\in[0,1]$, then $\mathcal{T}_{T,p}$ represented by (10) is continuous.

Proof: From (10), one has $\mathcal{T}_{T,p}(w,z)=[T(w^-,z^-),\max\{T(w^-,z^+),T(w^+,z^-),T(T(p,w^+),z^+)\}]$. Take into account that max keeps the continuous property. It is not difficult to arrive at the conclusion. ∎

It is effortless to gain Lemma 5.3 and Lemma 5.4.

Lemma V.3: $|w\wedge r-z\wedge r|\leq|w-z|$, $|w\vee r-z\vee r|\leq|w-z|$, thereinto $w,z,r\in[0,1]$.

Lemma V.4: If $W\to R^+$ functions f,g are bounded, in which W is a nonempty set and $R^+=[0,+\infty)$, then it implies that$\sup_{w\in W}f(w)\vee\sup_{w\in W}g(w)\leq\sup_{w\in W}\{f(w)\vee g(w)\}$.

Lemma V.5: ([35]) If $f,g:W\to R$ are bounded functions, in which W is a nonempty set and R is the set of real number, then aiming at any $w\in W$, one has (i) $|\sup_{w\in W}f(w)-\sup_{w\in W}g(w)|\leq\sup_{w\in W}|f(w)-g(w)|$; (ii) $|\inf_{w\in W}f(w)-\inf_{w\in W}g(w)|\leq\sup_{w\in W}|f(w)-g(w)|$.

Theorem V.3: In view of the identical conditions of Theorem 4.1, if the t-norm T_1 is continuous, then the $\zeta(w,z)$-GSI method for FMP indicated by (18) is uniformly continuous in d_{UN}, and hence continuous in d_{UN}.

Proof: Denote $\Gamma(w,z)=\mathcal{T}^{(2)}_{T,p}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{Q}(z)),\zeta(w,z))$, then (18) is changed into $\mathcal{Q}^\diamond(z)=\sup_{w\in W}\{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w),\Gamma(w,z))\},\quad z\in Z$.

On the basis of Lemma 5.2, one has that $\mathcal{T}^{(1)}_{T,p}$ is continuous. Then $\mathcal{T}^{(1)}_{T,p}$ is uniformly continuous w.r.t. its first variable in $[0_L,1_L]$. Hence, in allusion to any $\varepsilon>0$, there exists $\delta_1>0$

letting $d_{UN}(\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma(w,z)),\ \mathcal{T}_{T,p}^{(1)}(\mathcal{P}_2(w),\Gamma(w,z))) < \varepsilon$ be effective for any $z \in Z$ if $d_{UN}(\mathcal{P}_1,\ \mathcal{P}_2) < \delta_1$ (thereinto $\mathcal{P}_1, \mathcal{P}_2 \in \Phi(W)$). We adopt $\varepsilon_2 = \frac{\varepsilon}{2}$. Apparently $\varepsilon > \varepsilon_2 > 0$ is right. As a consequence, there exists $\delta_2 > 0$ such that $d_{UN}(\mathcal{P}_1,\ \mathcal{P}_2) < \delta_2$ implies $d_{UN}(\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma(w,z)),\ \mathcal{T}_{T,p}^{(1)}(\mathcal{P}_2(w),\Gamma(w,z))) < \varepsilon_2$ ($z \in Z$), that is, the following inequality holds ($z \in Z$)

$$\sup_{w\in W}\Big\{\big|[\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma(w,z))]^- - [\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_2(w),\Gamma(w,z))]^-\big| \vee \big|[\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma(w,z))]^+ - [\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_2(w),\Gamma(w,z))]^+\big|\Big\} < \varepsilon_2.$$

Let $\mathcal{Q}_1, \mathcal{Q}_2$ be the $\zeta(w,z)$-GMin-solution for $\mathcal{P}_1, \mathcal{P}_2$, in turn. In what follows, we prove that there exists $\delta > 0$ such that $d_{UN}(\mathcal{Q}_1, \mathcal{Q}_2) < \varepsilon$ if $d_{UN}(\mathcal{P}_1, \mathcal{P}_2) < \delta$.

Here we adopt $\delta = \delta_2$. Assume that $d_{UN}(\mathcal{P}_1, \mathcal{P}_2) < \delta$. Hence the inequality mentioned above holds, and on the strength of Lemma 5.3, Lemma 5.4 along with Lemma 5.5, one has

$$\begin{aligned} d_{UN}(\mathcal{Q}_1, \mathcal{Q}_2) &= \sup_{z\in Z}\{|\mathcal{Q}_1^-(z) - \mathcal{Q}_2^-(z)| \vee |\mathcal{Q}_1^+(z) - \mathcal{Q}_2^+(z)|\} \\ &\le \sup_{z\in Z}\{\sup_{w\in W}\{|[\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma(w,z))]^- - [\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_2(w),\Gamma(w,z))]^-| \vee \\ &\quad |[\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma(w,z))]^+ - [\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_2(w),\Gamma(w,z))]^+|\}\} \\ &\le \sup_{z\in Z}\varepsilon_2 = \varepsilon_2 < \varepsilon. \end{aligned}$$

As a consequence, it follows from Definition 5.3 that the $\zeta(w,z)$-GSI method for FMP denoted by (18) is uniformly continuous in d_{UN}.

It is apparent to note that if f is uniformly continuous then it is continuous. For this reason, we derive that the $\zeta(w,z)$-GSI method for FMP represented by (18) is also continuous in d_{UN}. ■

Theorem V.4: (i) In view of the identical conditions of Theorem 3.1 (or Theorem 3.3), if the t-norm T_1 is continuous, then the basic GSI method for FMP denoted by (12) is uniformly continuous in d_{UN}, and hence continuous in d_{UN}.

(ii) In view of the identical conditions of Theorem 3.2 (or Theorem 3.3), if the t-norm T_1 is continuous, then the basic GSI method for FMP represented by (13) is uniformly continuous in d_{UN}, and hence continuous in d_{UN}.

Proof: In Theorem 5.3, we choose $\zeta(w,z) \equiv 1_L$ ($w \in W, z \in Z$), then the conclusion for (12) is obtained. In addition, (13) is a peculiar status of (12), the result can be analogously derived. ■

Theorem V.5: In view of the identical conditions of Theorem 4.1, if the t-norms T_1, T_2 are continuous and I_1 is left-continuous w.r.t. the second variable, then the hierarchical $\zeta(w,z)$-GSI method denoted by (19) is uniformly continuous in d_{UN}, and hence continuous in d_{UN}.

Proof: Take into consideration that I_1 is left-continuous w.r.t. the second variable, and I_1 meets (PR5), then I_1 is continuous w.r.t. the second variable. Afterwards, from Proposition 2.2, it is apparent to discover that $\mathcal{I}_{I_1,p}$ is continuous w.r.t. the second variable, and thus $\mathcal{I}_{I_1,p}$ is uniformly continuous w.r.t. its second variable in $[0_L, 1_L]$.

Owing to that $\mathcal{T}_{T,p}^{(1)}, \mathcal{T}_{T,p}^{(2)}$ are continuous, $\mathcal{T}_{T,p}^{(1)}, \mathcal{T}_{T,p}^{(2)}$ are uniformly continuous w.r.t. its first variable and also its second variable in $[0_L, 1_L]$.

Let $\mathcal{O}_1$ be the hierarchical $\zeta(w,z)$-GMin-solution w.r.t. $\mathcal{P}_1, \mathcal{Q}_1$, and $\mathcal{O}_2$ be the one w.r.t. $\mathcal{P}_2, \mathcal{Q}_2$. We offer the signs: $\mathcal{O}_{11}(y) = \sup_{z\in Z}\{\mathcal{T}_{T,p}^{(1)}(\mathcal{Q}_1(z), \mathcal{T}_{T,p}^{(2)}(\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)),\quad \zeta_2(w,z)))\}$, $\mathcal{O}_{12}(y) = \sup_{z\in Z}\{\mathcal{T}_{T,p}^{(1)}(\mathcal{Q}_2(z), \mathcal{T}_{T,p}^{(2)}(\mathcal{I}_{I_1,p}(\mathcal{Q}(z),\mathcal{O}(y)), \zeta_2(w,z)))\}$, $\Gamma_{11}(w,z) = \mathcal{T}_{T,p}^{(2)}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{O}_{11}(y)),\ \zeta_1(w,z))$, $\Gamma_{12}(w,z) = \mathcal{T}_{T,p}^{(2)}(\mathcal{I}_{I_1,p}(\mathcal{P}(w),\mathcal{O}_{12}(y)),\quad \zeta_1(w,z))$ ($w \in W,\ z \in Z,\ y \in Y$). Then one has from (19) that $\mathcal{O}_1(y) = \sup_{w\in W}\{\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma_{11}(w,z))\}$, $\mathcal{O}_2(y) = \sup_{w\in W}\{\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_2(w),\Gamma_{12}(w,z))\}$ ($y \in Y$).

We validate that aiming at any $\varepsilon > 0$, there exists $\delta > 0$ letting $d_{UN}(\mathcal{O}_1,\ \mathcal{O}_2) < \varepsilon$ be effective if $d_{UN}(\mathcal{P}_1,\ \mathcal{P}_2) < \delta$ and $d_{UN}(\mathcal{Q}_1,\ \mathcal{Q}_2) < \delta$ where $\mathcal{P}_1, \mathcal{P}_2 \in \Phi(W)$ and $\mathcal{Q}_1, \mathcal{Q}_2 \in \Phi(Z)$.

We denote $\mathcal{O}_3(y) = \sup_{w\in W}\{\mathcal{T}_{T,p}^{(1)}(\mathcal{P}_1(w),\Gamma_{12}(w,z))\}$ ($y \in Y$).

We take $\varepsilon_0 = \varepsilon/2$. Take into account that $\mathcal{T}_{T,p}^{(1)}$ is continuous, hence there is $\delta_0 > 0$ such that $d_{UN}(\mathcal{P}_1, \mathcal{P}_2) < \delta_0$ implies $d_{UN}(\mathcal{O}_2, \mathcal{O}_3) < \varepsilon_0$.

We select $\varepsilon_1 = \varepsilon/2$. Analogously there is $\delta_1 > 0$ leading to that $d_{UN}(\mathcal{O}_1, \mathcal{O}_3) < \varepsilon_1$ if $d_{UN}(\Gamma_{11}(w,z), \Gamma_{12}(w,z)) < \delta_1$.

We adopt $\varepsilon_2 = \delta_1$. Owing to that $\mathcal{T}_{T,p}^{(2)}$ is continuous, for such $\varepsilon_2 > 0$, there exists $\delta_2 > 0$ resulting in that $d_{UN}(\Gamma_{11}(w,z), \Gamma_{12}(w,z)) < \varepsilon_2$ if $d_{UN}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{O}_{11}(y)), \mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{O}_{12}(y))) < \delta_2$.

We utilize $\varepsilon_3 = \delta_2$. Because $\mathcal{I}_{I_1,p}$ is uniformly continuous w.r.t. its second variable in $[0_L, 1_L]$. Hence for such $\varepsilon_3 > 0$, there is $\delta_3 > 0$ causing that $d_{UN}(\mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{O}_{11}(y)), \mathcal{I}_{I_1,p}(\mathcal{P}(w), \mathcal{O}_{12}(y))) < \varepsilon_3$ if $d_{UN}(\mathcal{O}_{11}, \mathcal{O}_{12}) < \delta_3$.

We select $\varepsilon_4 = \delta_3$. On the strength of Theorem 5.3, one has that the $\zeta(w,z)$-GMin-solution represented by (18) is uniformly continuous in d_{UN}. Then for such $\varepsilon_4 > 0$, there is $\delta_4 > 0$ such that $d_{UN}(\mathcal{O}_{11}, \mathcal{O}_{12}) < \varepsilon_4$ if $d_{UN}(\mathcal{Q}_1, \mathcal{Q}_2) < \delta_4$.

All in all, $d_{UN}(\mathcal{Q}_1, \mathcal{Q}_2) < \delta_4$ means $d_{UN}(\mathcal{O}_1, \mathcal{O}_3) < \varepsilon_1$.

Afterwards we adopt $\delta = \min\{\delta_0, \delta_4\}$. Then $d_{UN}(\mathcal{P}_1, \mathcal{P}_2) < \delta$ and $d_{UN}(\mathcal{Q}_1, \mathcal{Q}_2) < \delta$ implies $d_{UN}(\mathcal{O}_1, \mathcal{O}_2) \le d_{UN}(\mathcal{O}_2, \mathcal{O}_3) + d_{UN}(\mathcal{O}_1, \mathcal{O}_3) < \varepsilon_0 + \varepsilon_1 = \varepsilon$. That is, the hierarchical $\zeta(w,z)$-GSI method denoted by (19) is uniformly continuous in d_{UN}. And hence it is continuous in d_{UN}. ■

Theorem V.6: (i) In view of the identical conditions of Theorem 3.1 (or Theorem 3.3), if the t-norms T_1, T_2 are continuous and I_1 is left-continuous w.r.t. the second variable, then the hierarchical basic GSI method represented by (15) is uniformly continuous in d_{UN}, and hence continuous in d_{UN}.

(ii) Based upon the identical conditions of Theorem 3.2 (or Theorem 3.3), if the t-norms T_1, T_2 are continuous and I_1 is left-continuous w.r.t. the second variable, then the hierarchical basic GSI method denoted by (16) is uniformly continuous in d_{UN}, and hence continuous in d_{UN}.

Proof: In Theorem 5.5, we select $\zeta_1(w,z) \equiv 1_L$ and $\zeta_2(w,z) \equiv 1_L$ ($w \in W, z \in Z$), then the conclusion for (15) is achieved. By the way, (16) is a particular status of (15), the result can be analogously gotten. ■

VI. Applications and Discussion

When there are n rules, (2) should be altered to:

From $\mathcal{P}_1 \Longrightarrow \mathcal{Q}_1, \mathcal{P}_2 \Longrightarrow \mathcal{Q}_2, \ldots, \mathcal{P}_n \Longrightarrow \mathcal{Q}_n$, and $\mathcal{P}^\diamond$,
Gain the outcome $\mathcal{Q}^\diamond$.

Then the total inference rule is often acquired with $\vee$ (see [15], [36]), viz., $TRR(w,z) = \vee_{i=1}^n \mathcal{I}_{I_1,p}(\mathcal{P}_i(w), \mathcal{Q}_i(z))$. As a consequence, (5) should be altered to:

$$TRR(w,z) \rightarrow_2 (\mathcal{P}^\diamond(w) \rightarrow_1 \mathcal{Q}^\diamond(z)) \geq \zeta(w,z). \tag{20}$$

From Theorem 4.1, it is analogous to find that the $\zeta(w,z)$-GMin-solution $\mathcal{Q}^\diamond(z)$ gained from (20) is as below ($z \in Z$):

$$\sup_{w\in W} \{\mathcal{T}^{(1)}_{T,p}(\mathcal{P}^\diamond(w), \mathcal{T}^{(2)}_{T,p}(TRR(w,z),\ \zeta(w,z)))\}. \tag{21}$$

Here I_{GD} and I_L are adopted. I_{GD} is an R-implication, meanwhile I_L is an R-implication and also an (S,N)-implication. Let I_1 be I_{GD}, and I_2 be I_L. The following examples exhibit the procedure of the $\zeta(w,z)$-GSI method.

Example VI.1: Let $W = \{w_1, w_2, w_3\}$ *where* $w_1 = 0.2, w_2 = 0.4, w_3 = 0.6$, $Z = \{z_1, z_2\}$ *in which* $z_1 = 0.8, z_2 = 1.0$, $Y = \{y_1, y_2\}$ *in which* $y_1 = 0.4, y_2 = 0.8$, $p = 0.9$. *In addition,* $\zeta_1(w,z) = [\frac{1.0-w+z}{2}, \frac{1.2-w+z}{2}]$, $\zeta_2(w,z) = [\frac{0.8-w+z}{2}, \frac{1.0-w+z}{2}]$. *The rules and inputs are as below:*

$\mathcal{P} = \frac{[0.3,0.4]}{w_1} + \frac{[0.4,0.5]}{w_2} + \frac{[0.8,0.9]}{w_3}$, $\mathcal{Q} = \frac{[0.2,0.4]}{z_1} + \frac{[0.5,0.6]}{z_2}$,
$\mathcal{O} = \frac{[0.5,0.6]}{y_1} + \frac{[0.3,0.4]}{y_2}$,
$\mathcal{P}^\diamond = \frac{[0.1,0.2]}{w_1} + \frac{[0.2,0.3]}{w_2} + \frac{[0.7,0.8]}{w_3}$, $\mathcal{Q}^\diamond = \frac{[0.4,0.6]}{z_1} + \frac{[0.7,0.9]}{z_2}$.

We obtain by (19) that $\zeta(w,z)$*-GMin-solution is* $\mathcal{O}^\diamond = \frac{[0.3,0.4]}{y_1} + \frac{[0.2,0.3]}{y_2}$.

When $\rightarrow_1, \rightarrow_2$ employ the identical one, the GSI method degenerates to the interval-valued fully implicational method [13]. Homoplastically the interval-valued fully implicational method can also be split into the basic interval-valued fully implicational method and the $\zeta(w,z)$-interval-valued fully implicational method. Entire achievement of the GSI method (including its hierarchical mechanism) can also be extended to the corresponding interval-valued fully implicational method. Notice that the interval-valued fully implicational method in [13] is merely corresponding to the basic interval-valued fully implicational method.

Example VI.2: In allusion to the identical $\mathcal{P}, \mathcal{Q}, \mathcal{O}, \mathcal{P}^\diamond, \mathcal{Q}^\diamond, \zeta_1, \zeta_2$ *as Example 6.1, we adopt the hierarchical* $\zeta(w,z)$*-GSI method where* I_1, I_2 *take* I_{GD} *(that is the status of the interval-valued* $\zeta(w,z)$*-fully implicational method). By calculation, the optimal solution is* $\mathcal{O}^\diamond = \frac{[0.6,0.7]}{y_1} + \frac{[0.4,0.6]}{y_2}$.

Example VI.3: We show an example of emotion deduction in the field of affective computing [37]. Aiming at the acknowledged eight fundamental emotions (which are surprise, anxiety, expect, sorrow, angry, hate, joy, love), the former six emotions own a clear relationship with fury (as a novel emotion). We construct the emotion deduction system from six fundamental emotions to fury. Let $W = \{w_1, w_2, \ldots, w_6\}$ *where* $w_1 = 0, w_2 = 0.2, w_3 = 0.4, w_4 = 0.6, w_5 = 0.8, w_6 = 1.0$, *and* $Z = \{z_1\}$ *where* $z_1 = 0.6$, $p = 0.9$. *In addition,* $\zeta(w,z) = [\frac{1.0-w^2+z}{2}, \frac{1.2-w^2+z}{2}]$. *Some rules from* $\mathcal{P}_i$ *to* $\mathcal{Q}_i$ *and the input* $\mathcal{P}^\diamond$ *are as below:*

$\mathcal{P}_1 = \frac{[0.2,0.4]}{w_1} + \frac{[0.3,0.4]}{w_2} + \frac{[0.8,0.9]}{w_3} + \frac{[0.2,0.3]}{w_4} + \frac{[0.3,0.5]}{w_5} + \frac{[0.0,0.1]}{w_6}$,
$\mathcal{P}_2 = \frac{[0.6,0.8]}{w_1} + \frac{[0.5,0.6]}{w_2} + \frac{[0.4,0.5]}{w_3} + \frac{[0.7,0.8]}{w_4} + \frac{[0.6,0.7]}{w_5} + \frac{[0.5,0.7]}{w_6}$,
$\mathcal{P}_3 = \frac{[0.1,0.2]}{w_1} + \frac{[0.8,0.9]}{w_2} + \frac{[0.2,0.3]}{w_3} + \frac{[0.4,0.5]}{w_4} + \frac{[0.9,1.0]}{w_5} + \frac{[0.2,0.4]}{w_6}$,
$\mathcal{Q}_1 = \frac{[0.1,0.2]}{z_1}$, $\mathcal{Q}_2 = \frac{[0.4,0.5]}{z_1}$, $\mathcal{Q}_3 = \frac{[0.7,0.8]}{z_1}$,
$\mathcal{P}^\diamond = \frac{[0.4,0.5]}{w_1} + \frac{[0.9,1.0]}{w_2} + \frac{[0.3,0.4]}{w_3} + \frac{[0.0,0.1]}{w_4} + \frac{[0.5,0.6]}{w_5} + \frac{[0.8,0.9]}{w_6}$.

This is an example belonging to fuzzy classification, where three categories are $\mathcal{Q}_1(z_1) = [0.1, 0.2]$, $\mathcal{Q}_2(z_1) = [0.4, 0.5]$, $\mathcal{Q}_3(z_1) = [0.7, 0.8]$. *Let* I_1 *utilize* I_{GD}, *and* I_2 *be* I_L *in the* $\zeta(w,z)$*-GSI method. We calculate by (21) that the* $\zeta(w,z)$*-GMin-solution is* $\mathcal{Q}^\diamond(z_1) = \sup\{\xi(w_1,z_1), \xi(w_2,z_1), \ldots, \xi(w_6,z_1)\} = [0.4, 0.5] \vee [0.48, 0.58] \vee [0.3, 0.4] \vee [0.0, 0.1] \vee [0.18, 0.28] \vee [0.3, 0.4] = [0.48, 0.58]$. *It comes near to [0.4,0.5], and hence the classification outcome is the second class* $\mathcal{Q}_2$.

Example VI.4: Focusing on the identical $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{P}^\diamond, \zeta$ *as Example 6.3, we utilize the* $\zeta(w,z)$*-GSI method where* I_1, I_2 *take* I_{GD} *(that corresponds to the status of the* $\zeta(w,z)$*-interval-valued fully implicational method).*

Through calculation, one has that the $\zeta(w,z)$*-GMin-solution is as below:* $\mathcal{Q}^\diamond(z_1) = [0.4, 0.5] \vee [0.7, 0.8] \vee [0.3, 0.4] \vee [0.0, 0.1] \vee [0.48, 0.58] \vee [0.3, 0.4] = [0.7, 0.8]$. *It is the same as [0.7,0.8], and hence the classification outcome is the third class* $\mathcal{Q}_3$.

Example VI.5: In the field of the product conceptual design [38], a multi-functional traveling cup is taken as an example to describe the whole process of function tree modeling. The core functional requirements of this multi-functional traveling cup include four parts: heat preservation, non-breakable demand, portable demand and large capacity. Around these four functions, it is assumed that an initial fuzzy function tree as shown in Fig. 1 is obtained (in which G_1 *is an "AND" gate node).*

Here we reveal the requirements for heat preservation and non-breakable demand, which are two inseparable functions. The inputs include heat preservation and non-breakable demand. The outputs incorporate bilayer structure, insulation degree in the middle area, thickness of bilayer structure (the former

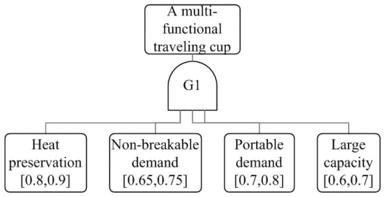

Fig. 1. An initial fuzzy function tree.

three are for heat preservation), bilayer structure, firmness of outer wall, thickness of bilayer structure (the latter three are for non-breakable demand). In detail, let $W = \{w_1, w_2\}$ *where* $w_1 = 0.3, w_2 = 0.2,$ *and* $Z = \{z_1, z_2, \ldots, z_6\}$ *where* $z_1 = 0.1, z_2 = 0.2, z_3 = 0.3, z_4 = 0.4, z_5 = 0.5, z_6 = 0.6,\ p = 0.9.$ *In addition,* $\zeta(w,z) = [\frac{2.0-w\vee z}{2}, \frac{2.2-w\vee z}{2}]$*. Some rules from* $\mathcal{P}_i$ *to* $\mathcal{Q}_i$ *and the input* $\mathcal{P}^\diamond$ *are as below:*

$\mathcal{P}_1 = \frac{[0.90,1.00]}{w_1} + \frac{[0.80,0.90]}{w_2},\ \mathcal{Q}_1 = \frac{[0.80,0.90]}{z_1} + \frac{[0.85,0.95]}{z_2} + \frac{[0.80,0.95]}{z_3} + \frac{[0.60,0.70]}{z_4} + \frac{[0.80,0.95]}{z_5} + \frac{[0.75,0.85]}{z_6},$

$\mathcal{P}_2 = \frac{[0.55,0.65]}{w_1} + \frac{[0.20,0.30]}{w_2},\ \mathcal{Q}_2 = \frac{[0.45,0.55]}{z_1} + \frac{[0.50,0.60]}{z_2} + \frac{[0.45,0.60]}{z_3} + \frac{[0.05,0.15]}{z_4} + \frac{[0.20,0.30]}{z_5} + \frac{[0.10,0.20]}{z_6},$

$\mathcal{P}_3 = \frac{[0.40,0.50]}{w_1} + \frac{[0.30,0.40]}{w_2},\ \mathcal{Q}_3 = \frac{[0.35,0.45]}{z_1} + \frac{[0.30,0.40]}{z_2} + \frac{[0.25,0.35]}{z_3} + \frac{[0.20,0.30]}{z_4} + \frac{[0.30,0.45]}{z_5} + \frac{[0.15,0.25]}{z_6},$

$\mathcal{P}_4 = \frac{[0.65,0.75]}{w_1} + \frac{[0.90,1.00]}{w_2},\ \mathcal{Q}_4 = \frac{[0.60,0.70]}{z_1} + \frac{[0.55,0.65]}{z_2} + \frac{[0.50,0.60]}{z_3} + \frac{[0.80,0.90]}{z_4} + \frac{[0.75,0.90]}{z_5} + \frac{[0.60,0.70]}{z_6},$

$\mathcal{P}^\diamond = \frac{[0.80,0.90]}{w_1} + \frac{[0.65,0.75]}{w_2}.$

Let I_1 *utilize* I_{GD}*, and* I_2 *be* I_L *in the* $\zeta(w,z)$*-GSI method. In accordance with the* $\zeta(w,z)$*-GSI method, we calculate the* $\zeta(w,z)$*-GMin-solution represented by (21). Then we obtain that the* $\zeta(w,z)$*-GMin-solution is* $\mathcal{Q}^\diamond = \frac{[0.65,0.85]}{z_1} + \frac{[0.7,0.85]}{z_2} + \frac{[0.65,0.85]}{z_3} + \frac{[0.8,0.9]}{z_4} + \frac{[0.75,0.85]}{z_5} + \frac{[0.45,0.55]}{z_6}.$

Meanwhile, if we utilize the $\zeta(w,z)$*-GSI method where* I_1, I_2 *take* I_{GD} *(that corresponds to the status of the* $\zeta(w,z)$*-interval-valued fully implicational method, then the* $\zeta(w,z)$*-GMin-solution is* $\mathcal{Q}^\diamond = \frac{[0.8,0.9]}{z_1} + \frac{[0.8,0.9]}{z_2} + \frac{[0.8,0.9]}{z_3} + \frac{[0.8,0.9]}{z_4} + \frac{[0.75,0.85]}{z_5} + \frac{[0.7,0.8]}{z_6}.$

After similar processes (by using $I_1 = I_{GD}$ *and* $I_2 = I_L$ *in the* $\zeta(w,z)$*-GSI method), the expanded tree is shown in Fig. 2. Among it,* G_4 *is an "OR" gate node, and* G_{11} *is an "OR-NOT" gate node, and the other nodes are similar. The final design schemes can be obtained through function solving of the function tree as shown in Fig. 2. Here we show one solution. In detail, we use a bilayer hollow structure to achieve the function of heat preservation, and adopt a retractable cup bladder and a retractable cup body (resulting in that corresponding volume is variable), and add a steel bottom cover outside (to ensure that it is not easy to break), and employ a strap outside the body.*

Here we exhibit some comparative analyses. The foregoing symmetric implicational principles are direct at the generic fuzzy sets. In the granular symmetric implicational (GSI) principles, the computing procedure and the gained outcome are all related to interval-valued fuzzy sets. These are in more excellent manner than the corresponding situations from generic fuzzy sets.

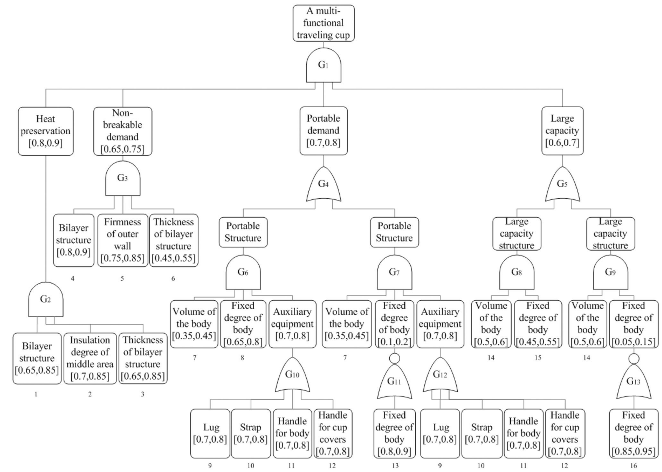

Fig. 2. An initial fuzzy function tree.

Accordingly, these GSI principles ameliorate the previous ones of the symmetric implicational method.

Compared with the $\zeta(w,z)$-interval-valued fully implicational method, the advantages of the proposed GSI method are reflected in the following three aspects. (i) The GSI method makes the inference appear more coherent. Example 6.1 and Example 6.2 direct at the identical rule base and input, and then the optimal solution to the $\zeta(w,z)$-interval-valued fully implicational method in Example 6.2 is bigger than the one of the $\zeta(w,z)$-GSI method in Example 6.1. On the basis of the intrinsic spirit of the $\zeta(w,z)$-GSI method (viz., the $\zeta(w,z)$-GSI principle which tries to gain the smallest solution letting (17) be effective), the $\zeta(w,z)$-GSI method offers better solution, and makes the inference more compact (which is a vital evaluating standard for the fuzzy inference strategy). In the meantime, we can obtain that the basic GSI method has similar advantages than the basic interval-valued fully implicational method. Similarly, from Example 6.3, Example 6.4 and Example 6.5, the same conclusion can be obtained. (ii) There exists a step response phenomenon in the $\zeta(w,z)$-interval-valued fully implicational method. In Example 6.5, the same output (i.e., [0.8,0.9]) occurs four times out of six values (for different inputs). This is clearly not what we want. However, the GSI method solves this problem properly by using two implications. (iii) The application effect of the GSI method is better. For Example 6.3 and Example 6.4, the more reasonable granular inference result is the second class $\mathcal{Q}_2$ in accordance with expert knowledge in the field of affective computing. For Example 6.5, if the value of the child node (that realizes a conceptual design requirement in the function tree modeling) is smaller, then it indicates that the requirement can be more easily implemented and that the established function tree is superior. Consequently, the $\zeta(w,z)$-GSI method all achieves more reasonable result than the $\zeta(w,z)$-interval-valued fully implicational method.

There are some works related to granular computing (e.g. [18], [19]). In this study, we systematically establish a granular structure in the scope of fuzzy inference, which is a novel exploration for the field of granular computing. It is of great value for granular expression, modeling and inference. So this is an important impetus to the research of granular computing.

VII. Conclusions

We come up with and investigate the granular symmetric implicational (GSI) method, which incorporates the basic GSI method and the $\zeta(w,z)$-GSI method as well as their hierarchical mode.

The contributions of the study are concisely captured as follows. In the first place, we offer a new construction method for interval-valued implications as well as related adjoint couples, which is gained from some conditions of implication on [0,1]. Here we adopt complete residuated lattices as the structures of truth-values for interval-valued fuzzy sets. In the second place, unified expressions of optimal solutions to the basic GSI method and the $\zeta(w,z)$-GSI method are acquired, in which $\rightarrow_1, \rightarrow_2$ utilize R-implications or (S, N)-implications. Afterwards, the optimal solutions to these methods are acquired for eleven specific implications, which cover $I_{GG}, I_F, I_L, I_{KD}, I_R, I_{GD}, I_E, I_D, I_Y, I_{MK}, I_{TD}$. In the third place, in allusion to multiple rules, we set up hierarchical inference strategy for these methods and gain the corresponding hierarchical solutions.

The connotation of "granules" in the GSI method includes three points. To begin with, $P, Q, P^\diamond, Q^\diamond$ all fall into the category of information granules. They utilize interval-valued fuzzy sets as their specific forms. Furthermore, $\rightarrow_1$ and $\rightarrow_2$ can adopt different interval-valued implications. Thereinto $\rightarrow_1$ gives expression to the implication connective in a logic system and $\rightarrow_2$ indicates the "if-then" relation of the inference model (1). $\rightarrow_1$ and $\rightarrow_2$ both reflect the mapping relationship of information granules. Lastly, aiming at the status of multiple rules, we make use of the hierarchical granular mode to carry through inference. In short, the granular structure is constructed in the GSI method.

The advantages of the proposed method are three-fold. First, new symmetric implicational principles are presented, which gain an advantage over the previous ones, because the interval-valued fuzzy sets offer more powerful expression abilities than generic fuzzy sets. Second, the reversibility properties of the basic GSI method are validated, and the continuous and uniformly continuous properties of the GSI method in d_{UN} are proved. Third, from two specific operation examples and two emotion deductive examples, it is discovered that the GSI method is superior over the corresponding interval-valued fully implicational method, owing to that the GSI method makes the inference appear more coherent.

In future studies it would be of interest to construct fuzzy systems based upon the GSI method, which also incorporate fuzzier, defuzzier. And the corresponding performances including universal approximation, response abilities and so forth would be discovered. In addition, it is also worth combining the GSI method with some fuzzy clustering algorithms (e.g., the patch-based fuzzy local similarity c-means algorithm in [39]) to form new clustering framework under the environment of granular computing.

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